NOTES

Serial Isogons of 90 Degrees

Lee– At last! Martin

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In 1988 the first author devised a computer program to search a unit-square grid for closed paths with the following properties. The path starts along a lattice line with a segment of unit length, turns 90 degrees in either direction, continues for 2 units, turns again in either direction, continues for 3 units, and so on. In other words, the segments of the path are in serial order $1, 2, 3, \ldots N$, with a right angle turn at the end of each segment. A path of N segments—the number is of course the same as the number of turns or corners—is said to be a path of order N.

If the path returns to its starting point, making a right-angle with its first segment, we call it a *serial isogon* of 90 degrees. The isogon is allowed to self-intersect, to touch at corners, and to overlap along segments. Think of it as a serial walk through a city of square blocks and returning to the starting corner, or as the moves of a rook on a large enough chessboard.

It is not obvious that such paths exist. However, with a little doodling you will discover the unique isogon of order 8, the lowest order a 90-degree isogon can have. It outlines a polyomino of 52 unit squares that, as FIGURE 1 shows, tiles the plane. Indeed, it satisfies the "Conway criterion" [1] for identifying tiling shapes. The boundary of the polyomino can be partitioned into six parts, the first and fourth of which (AB and ED in FIGURE 2) are equal and parallel, while the other four ($BC = 3$; $CD = 4, 5, 4$; $EF = 7$; $FA = 1, 8, 1$) each have rotational symmetry through 180° about their midpoints (black dots in FIGURE 2). It may be the only plane-tiling polyomino with a serial boundary, but we are unable to prove this.

Puzzle: Can you tile this polyomino with thirteen L-shaped tetrominoes?

It is easy to prove that N must be a multiple of 4. One way is to consider N rook moves on a bicolored chessboard. Assume without loss of generality that the rook begins on a black cell and makes its first move horizontally. To close the path, its final move must be vertical, and end on a black square. Because moves of odd and even length alternate, the sequence of colors at the end of each segment forms the repeating sequence: $WW\ BB\ WW\ BB \ldots$. The rook returns to a black cell, after a vertical move, if and only if the number of moves $\equiv 0 \bmod 4$.

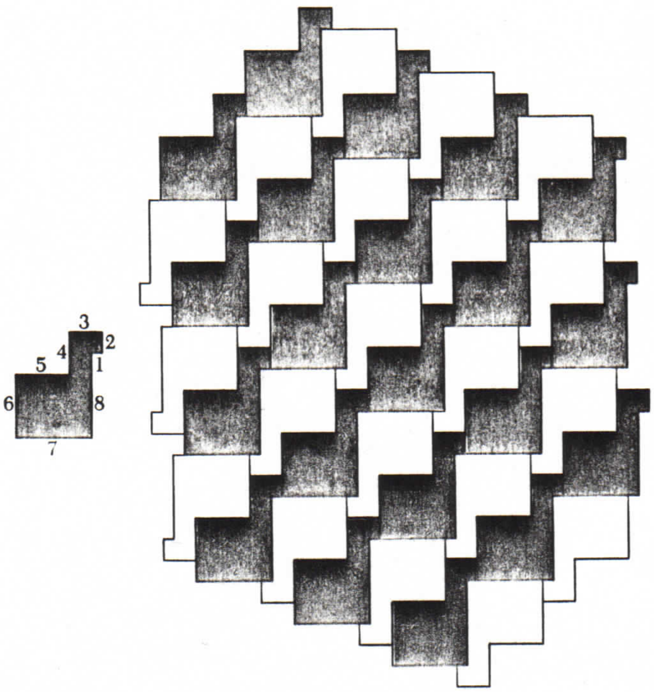

FIGURE 1
The only known serial-sided polyomino that tiles the plane.

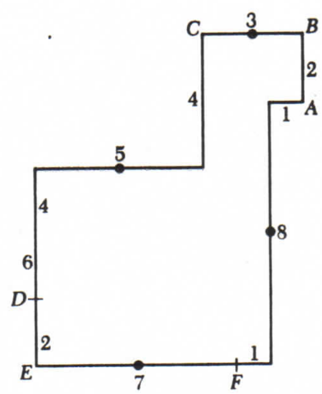

FIGURE 2
Applying Conway's criterion to prove that the polyomino tiles the plane.

Experimenting on graph paper will quickly convince you that there is no serial isogon of order 4. (You might call this "a 4-gon conclusion".) We have exhibited one of order 8. After the computer program exhaustively plotted all serial isogons of orders through 24, a surprising fact emerged. No serial isogons were found except when N is a multiple of 8. This led to several proofs of the following theorem:

For any 90-*degree serial isogon*, N *must be a multiple of* 8.

Assume that a closed path begins with a unit move to the east, and that moving east or north is positive, and moving south or west is negative. A path can be uniquely described by placing a plus or minus sign in front of each number in the sequence of moves to indicate the direction of the move. For example, the order-8 polyomino has the following formula:

$$+1+2-3-4-5-6+7+8.$$

It is obvious that if the path closes, the sum of all horizontal moves—the odd numbers—must be zero, otherwise the path will not return to the vertical lattice line that goes through the starting point. Similarly, the sum of all vertical moves—even numbers—must be zero or the path will not return to the horizontal lattice line going through the origin point. The sum of all the numbers will, of course, also be zero.

We know that N is a multiple of 4, say $4k$. Then the north-south moves are the even-length ones, $2, 4, \dots, 4k$. The total north-south distance is therefore $2(1+2+\cdots+2k) = 2k(2k+1)$. Half of this, $k(2k+1)$, must be north and half of it south. But if k is odd, this distance is odd, and cannot be the sum of even-length moves.

We can make this clearer by taking $N = 12$ as an example. Even numbers in this path's formula $(2, 4, 6, 8, 10, 12)$ add to 42. If the formula describes a closed path, the sum of the positive numbers in this sequence must equal $42/2 = 21$. But no set of even numbers can add to the odd number 21. Consequently, no formula can be constructed that will describe a closed path of order 12.

With reference to the grid, this tells us that if N is a multiple of 4 but not of 8, the last segment of the path, which is vertical, cannot return to the horizontal lattice line that goes through the path's origin point because the sum of the positive segments going north cannot equal the absolute sum of the negative segments going south. The path's end will always be an even number of units above or below the zero horizontal line.

Is there a coloring pattern that proves the $N \equiv 0 \bmod 8$ theorem? Yes, the simple coloring shown in FIGURE 3 (found by the second author) will do the trick.

Start a path on any black cell, then make your first move horizontally in either direction. Regardless of your choices of how to turn at the end of each segment, the colors at the ends of segments will endlessly repeat the sequence: *WWWWWWBB*,

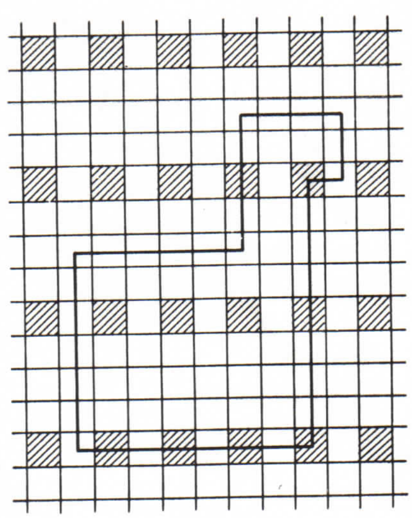

FIGURE 3

A bicoloring of the grid for proving that N is a multiple of 8 for all orders of closed serial paths. The order-8 polyomino is shown in outline.

WWWWWWBB, *WWWWWWWBB*,... . The path will enter a black cell (which it must do if the path closes) in a direction perpendicular to the first segment if and only if the number of moves is a multiple of 8. Shown in the illustration is the path that outlines the order-8 tiling polyomino.

Although all closed paths have formulas in which the total sum of the signed terms is zero, this is not sufficient to produce a formula for a path. A serial path closes if and only if the even numbers in its formula add to zero, and likewise the odd numbers. (Nonserial closed paths may meet this proviso, and be of any order of 4 or greater that is a multiple of 2. The formula $+1+2-1-2$, for example, describes a closed path that outlines a domino.) If this proviso is not met, the formula gives the location of the path's final point with respect to the origin. If the sum of the signed even numbers is positive, it gives the number of units where the path ends above the origin; if negative, it gives the number of units below the origin. Similarly for the sum of the signed odd numbers. If positive, it gives the end point's distance east of the origin; if negative, the distance west.

If the signs of all even numbers are changed, or if the signs of all odd numbers are changed, it reflects the isogon along an orthogonal axis. If all signs are changed, it rotates the isogon 180 degrees.

We have shown that $N \equiv 0 \bmod 8$ is a necessary condition for a closed serial path. Is it also sufficient? Yes. Here is one way to arrange plus and minus signs in a formula that will always describe a serial isogon: Put plus signs in front of the first and last pairs of numbers. Put minus signs in front of the next to last pairs of numbers at each end, and continue in this way until all pairs of numbers are signed. This ensures that all even numbers add to zero, and likewise all odd numbers, therefore the formula must describe a closed path. It produces, for instance, the unique formula for $N = 8$. Applied to $N = 16$ it gives the formula $+1+2-3-4+5+6-7-8-9-10+11+12-13-14+15+16$, which describes the isogon at position O_2E_5 in FIGURE 4.

Is there a procedure guaranteed to construct a serial isogon for any order $N \equiv 0 \bmod 8$ that outlines a polyomino? The answer is again yes. Each formula has $8n$ numbers. If we make positive all numbers in the first fourth of the formula, and also in the last fourth, and make negative all numbers in the half in between, we produce a serial isogon. Applied to $N = 16$, it gives $+1+2+3+4-5-6-7-8-9-10-11-12+13+14+15+16$, which describes the polyomino at O_1E_4 in FIGURE 4. The polyomino generated by this procedure always takes the form of a snake that grows longer as N increases. FIGURE 5 shows the polyomino snake for order 32. A diagonal line, running from the extreme corners of the snake's head and tail, going between all the interior corners, is almost, but not quite, straight.

We now turn to the more difficult task of enumerating all possible serial isogons (not counting rotations and reflections as different) for a given isogon. As mentioned earlier, there is only one isogon of order 8, the tiling polyomino. For $N = 16$, the computer program found the 28 solutions shown in FIGURE 4. Note that only three (O_1E_1, O_1E_4, O_1E_7) are polyominoes. For $N = 24$, the program produced 2,108 distinct isogons, of which 67 bound polyominoes. For $N = 32$ the program's running time became too long to be feasible.

No formula is known for enumerating all distinct serial isogons of order N, or for counting the polyominoes of a given order. However, there are procedures by which the number of isogons can be counted by hand to a value of N that goes well beyond $N = 24$.

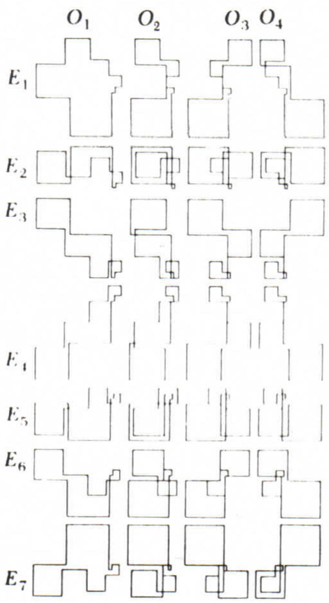

FIGURE 4
The 28 distinct order-16 serial isogons of 90°.

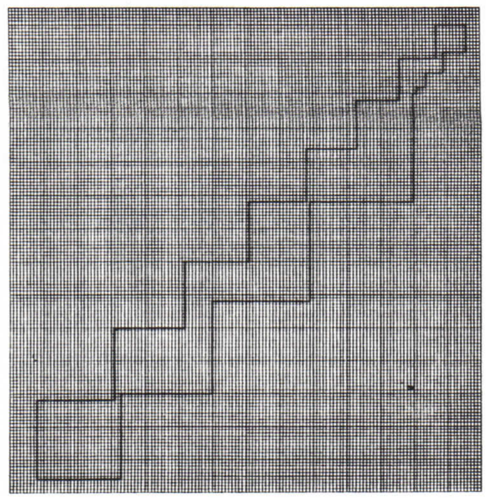

FIGURE 5
The snake polyomino of order 32.
The diagonal line separating the snake's two sides is almost, but not quite, straight.

Here is how the fourth author describes one such procedure:

Suppose $N = 16$; it's easy to see that this quantity is the constant term if you expand the algebraic product

$$(x^{-1}+x^{1})(y^{-2}+y^{2})(x^{-3}+x^{3})\cdots(x^{-15}+x^{15})(y^{-16}+y^{16})$$

into powers of x and y. To get the number of ways for the odd sum to cancel, we want the constant term of

$$(x^{-1}+x^{1})(x^{-3}+x^{3})\cdots(x^{-15}+x^{15}),$$

which is the coefficient of $x^{1+3+5+\cdots+15}=x^{64}$ in

$$(1+x^{2})(1+x^{6})\cdots(1+x^{30}),$$

which is the coefficient of x^{32} in

$$(1+x)(1+x^{3})\cdots(1+x^{15}),$$

which is 8. To get the number of ways for the even sum to cancel, we want the constant term of

$$(y^{-2}+y^{2})(y^{-4}+y^{4})\cdots(y^{-16}+y^{16}),$$

which is the constant term of

$$(y^{-1}+y^{1})(y^{-2}+y^{2})\cdots(y^{-8}+y^{8}),$$

which is the coefficient of $y^{1+2+\cdots+8}=y^{36}$ in

$$(1+y^{2})(1+y^{4})\cdots(1+y^{16}),$$

which is the coefficient of y^{18} in

$$(1+y)(1+y^2)\cdots(1+y^8),$$

which is 14. So the total number of closed paths is 8×14; divide by 4 to get 28 closed paths that are distinct under reflectional symmetry. Suppose we start with $+1+2$; then the four ways to do the odd numbers are

$$O_1 = +1+3-5-7-9-11+13+15$$
$$O_2 = +1-3+5-7-9+11-13+15$$
$$O_3 = +1-3-5+7+9-11-13+15$$
$$O_4 = +1-3-5+7-9+11+13-15$$

and the seven ways to do the even numbers are

$$E_1 = +2+4+6+8-10-12-14+16$$
$$E_2 = +2+4+6-8+10-12+14-16$$
$$E_3 = +2+4-6+8+10+12-14-16$$
$$E_4 = +2+4-6-8-10-12+14+16$$
$$E_5 = +2-4+6-8-10+12-14+16$$
$$E_6 = +2-4-6+8+10-12-14+16$$
$$E_7 = +2-4-6+8-10+12+14-16.$$

The three serial polyominoes are the snake O_1E_4 and two other solutions O_1E_1, O_1E_7. (It's curious that only O_1 can be completed. The case O_2E_1 almost works, but that path gives a degenerate polyomino whose width is zero at one point. Paths O_3E_1, O_3E_4, and O_4E_4 fail in the same way.)

In general when $N = 8n$, the number of closed paths is the product of the coefficient of x^{8n^2} in

$$(1+x)(1+x^3)(1+x^5)\cdots(1+x^{8n-3})(1+x^{8n-1})$$

and the coefficient of y^{4n^2+2} in

$$(1+y)(1+y^2)(1+y^3)\cdots(1+y^{4n-1})(1+y^{4n}).$$

These numbers, for small n (divided by 2 to remove symmetry), are

N	n	odds/2	evens/2	product
8	1	1	1	1
16	2	4	7	28
24	3	34	62	2108
32	4	346	657	227322
40	5	3965	7636	30276740
48	6	48396	93846	4541771016
56	7	615966	1199892	739092675672
64	8	8082457	15796439	127674038970623

It seems certain that the vast majority of these isogons will not bound polyominoes. The paper of Bhattacharya and Rosenfeld [3] is concerned with the problem of avoiding self-intersections in isogons: they treat the general problem in which the sides are of arbitrary length, not just our particular case of consective integers.

Here is how the third author has made an asymptotic estimate of the number of serial isogons:

We have seen that the number of isogons of a given order is the product of half the number of possible choices of sign in $\pm 2 \pm 4 \pm 6 \pm \cdots \pm (8n-2) \pm 8n = 0$ with half the number of choices of sign in $\pm 1 \pm 3 \pm 5 \pm \cdots \pm (8n-3) \pm 8n = 0$.

The first of these is the number of partitions of half of the sum

$$2 + 4 + 6 + \cdots + 8n$$

into distinct even parts, of size at most $8n$, i.e., the number of partitions of $n(4n+1)$ into d distinct parts of size $\le 4n$. Subtract $1, 2, \ldots, d$ from these parts, now no longer necessarily distinct, of size $\le 4n - d$. We require the number of partitions of $4n^2 + n - (1/2)d(d+1)$ into at most d parts, no longer necessarily distinct, of size $\le 4n - d$. Here d, the number of parts, lies in the approximate range $(4 - 2\sqrt{2})n < d < 2\sqrt{2}\,n$.

In the same way the second number is the number of partitions of $4n^2 - (1/2)d^2$ into at most d parts, not necessarily distinct, of size $\le 4n - d$, with d in approximately the same range as before, but with d necessarily even!

The main contribution comes from $d = 2n$ and the distribution, as we shall see, is essentially the binomial distribution, so that a good estimate of the whole is obtained by multiplying this central term by $\sqrt{4\pi n}$, except that we halve the "odd" estimate since only alternate terms (d even) are taken.

From formula (75) in [2] we learn that the number of partitions of j into at most a parts, with each part $\le b$, is asymptotically equal to

$$\frac{1}{\sigma_{a,b}}\binom{a+b}{a}\phi\left(\frac{j - ab/2}{\sigma_{a,b}}\right),$$

where $\sigma_{a,b}^2 = ab(a+b+1)/12$ and $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$.

In both the even and the odd cases, $a = d$, $b = 4n - d$, $a + b + 1 = 4n + 1$, and the central term is given by $d = 2n = a = b$ for which $j = 4n^2 + n - n(2n+1)$ and $j = 4n^2 - 2n^2$, i.e., $2n^2$ in either case, and $j - ab/2 = 0$, so that the central term is asymptotically equal to

$$\frac{1}{\sqrt{2\pi}}\,\frac{\sqrt{3}}{2n\sqrt{4n+1}}\binom{4n}{2n},$$

i.e., asymptotically equal to

$$\frac{\sqrt{3}}{2n\sqrt{2\pi}\sqrt{4n+1}}\,\frac{2^{4n}}{\sqrt{4\pi n}}$$

by Stirling's formula. Multiply by $\sqrt{4\pi n}$ to estimate the total number of partitions in the even case, and by half of that in the odd case. The product of the halves of these numbers (i.e., not counting the E.-W. or N.-S. reflections of the isogons as different) is thus

$$\frac{1}{8}\left\{\frac{\sqrt{3}}{2n\sqrt{2\pi}\sqrt{4n+1}}2^{4n}\right\}^2 = \frac{3 \cdot 2^{8n-6}}{\pi n^2(4n+1)}.$$

Compare this estimate with the actual values obtained above.

n	$3 \times 2^{8n-6}/\pi n^2(4n+1)$
1	0.76
2	27.2
3	2140
4	235604
5	31248698
6	4666472281
7	756618728785
8	130321844073100

The first author has since explored serial isogons on isometric grids. They are of two types: those with 60-degree angles, and those with 120-degree angles. In the 60-degree case, serial isogons exist if and only if $N \geq 9$ and $N \not\equiv 1 \bmod 3$. In the 120-degree case, they exist if and only if N is a multiple of 6. If each angle of a serial polygon can be either 60 or 120 degrees, such polygons exist for any N that is 5 or greater. In all three cases, the smallest order is unique and is a *polyiamond*—a polygon formed by joining unit equilateral triangles. The three are shown in FIGURE 6. Note that the order-5 polyiamond tiles the plane in two different ways, see [4].

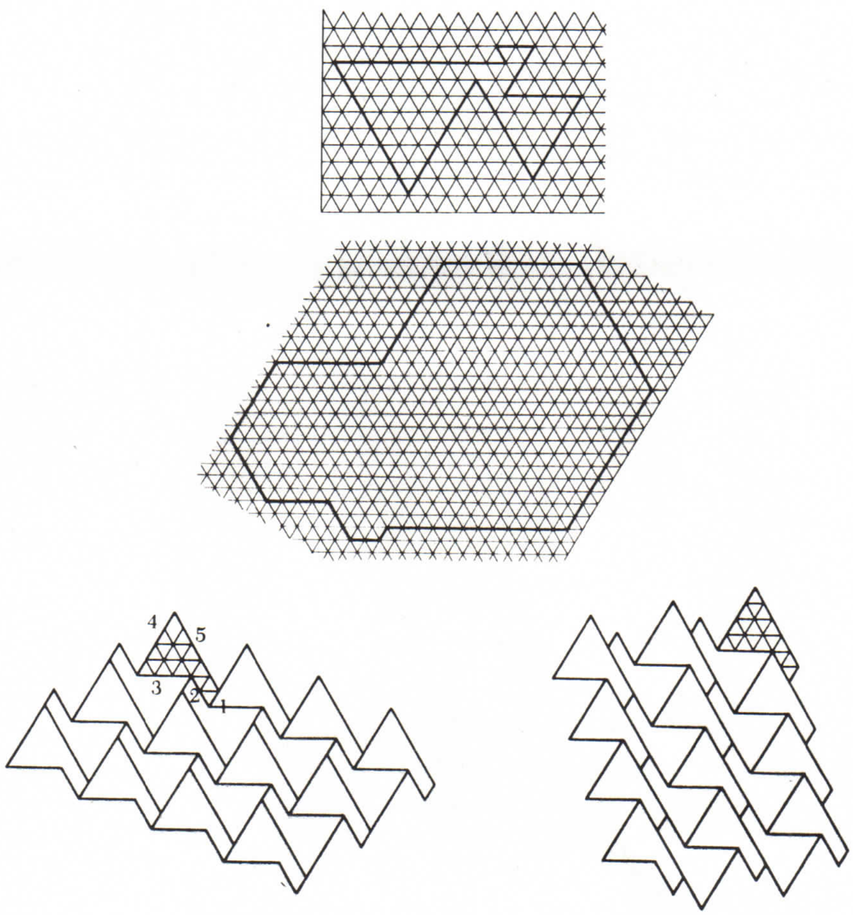

FIGURE 6

The unique, smallest examples of serial isometric isogons. At top is the only order-9 isogon with 60-degree angles. In the middle is the only order-12 isogon with 120-degree angles. At bottom, shown tiling the plane in two ways, is the only order-5 isogon that mixes the two angles.

For a time the first author believed that closed serial paths existed only when angles are 60, 90, or 120 degrees, but then he suddenly discovered such a path with 108-degree angles, the angles of a regular pentagon. More recently Hans Cornet, a retired mathematician in The Hague, has put forward a proof that at least one serial isogon can be constructed for every angle α that is a rational multiple of 360 degrees, that is, when $\alpha = 2\pi n/m$ radians, with m and n positive integers; see [4].

The first author has also investigated closed paths on square and isometric grids that have segment lengths in sequences other than the counting numbers, such as Fibonacci sequences, consecutive primes, and so on. He has produced a whimsical class of *piominoes*—polyominoes whose sides in cyclic order are the first n digits of pi, with zeros omitted. Because the digits of pi are pseudo-random, the task of enumerating pi-isogons of 90 degrees is related to determining the probability of self-intersecting random walks on a square lattice. He has even experimented with closed paths based on the letters of number words. An example is shown in FIGURE 7. It is unquestionably one of the most useless polyomino outlines ever constructed, yet does it not have a curious charm?

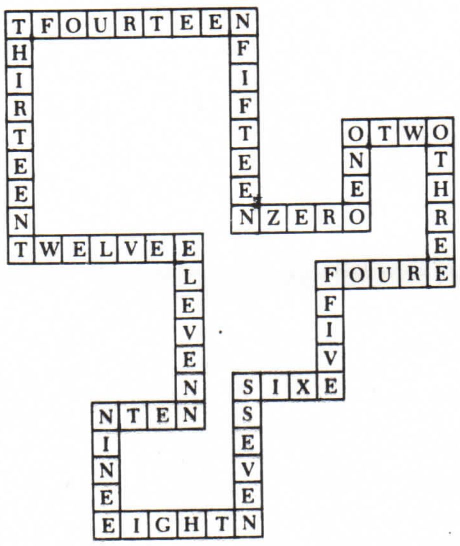

FIGURE 7

A 16-sided polyomino, its side determined by the words for zero through 15.

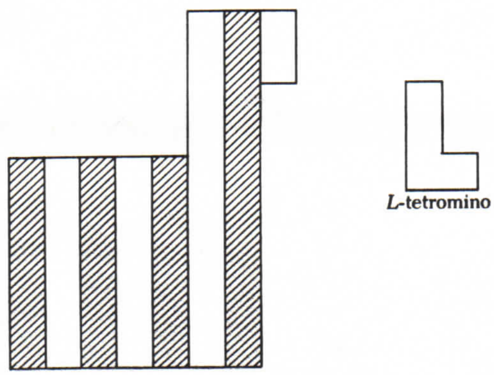

FIGURE 8

A coloring pattern on the tiling polyomino for proving it cannot be dissected into 13 *L*-tetrominoes.

Solution to puzzle: To prove that 13 *L*-tetrominoes will not tile the polyomino, divide the polyomino into alternatively colored vertical stripes as shown in FIGURE 8. No matter how an *L*-tetromino is placed within this pattern, it will cover an odd number of cells of each color. Thirteen odd numbers add to an odd number, but the polyomino has an even number of cells of each color. The tiling is therefore impossible.

Postscript

The appearance of a prepublication version of this paper [4] has resulted in a good deal of correspondence and misunderstanding. John Leech observes that the argument there, showing that N is a multiple of 8, is faulty and exacerbated by the statement that "even numbers hardly entered into the discussion," whereas we know that it is the even lengths that clinch the matter. Leech also gives an argument involving those gridlines that "quarter" a tetromino, which is equivalent to that of FIGURE 8. The difficulty of printing a complicated formula in a popular article resulted in an oversimplification that was no longer an asymptotic formula in the sense described. Finally, it was implied that calculation of the exact numbers of serial isogons was more difficult than is actually the case, which prompted several people to reach for their computers. Ilan Vardi [6] has explained how to compute the numbers rapidly via the Chinese Remainder Theorem, and he lists the values for $N = 400$ and $N = 1000$. Calculations up to $N = 200$ were carried out by Sivy Farhi, 815 S. California #B, Monrovia, CA and by Pierre Barnouin, Chemin de Stramousse, 06530 Cabris, France, who gave the following values:

16	28
24	2108
32	227322
40	30276740
48	4541771016
56	739092675672
64	127674038970623
72	23085759901610016
80	4327973308197103600
88	835531767841066680300
96	165266721954751746697155
104	33364181616540879268092840
112	6854017416098227836106023048
120	1429368258586343246184813682344
128	302023498629081603279538134332922
136	64557914743374337032608546756101824
144	13941247125893997584457711273087122310
152	3038225349257507092516361163813831321438
160	667575475791956832191676953455074834982100
168	147773788473936923724715382248726990582150405
176	32931659242107964657022264548538525142956914056
184	7383987729780296585063944629621065123001927478725
192	1664961555710273709724126262313969341483976633058164
200	377359709872056562198423857053288570232577607987443492

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