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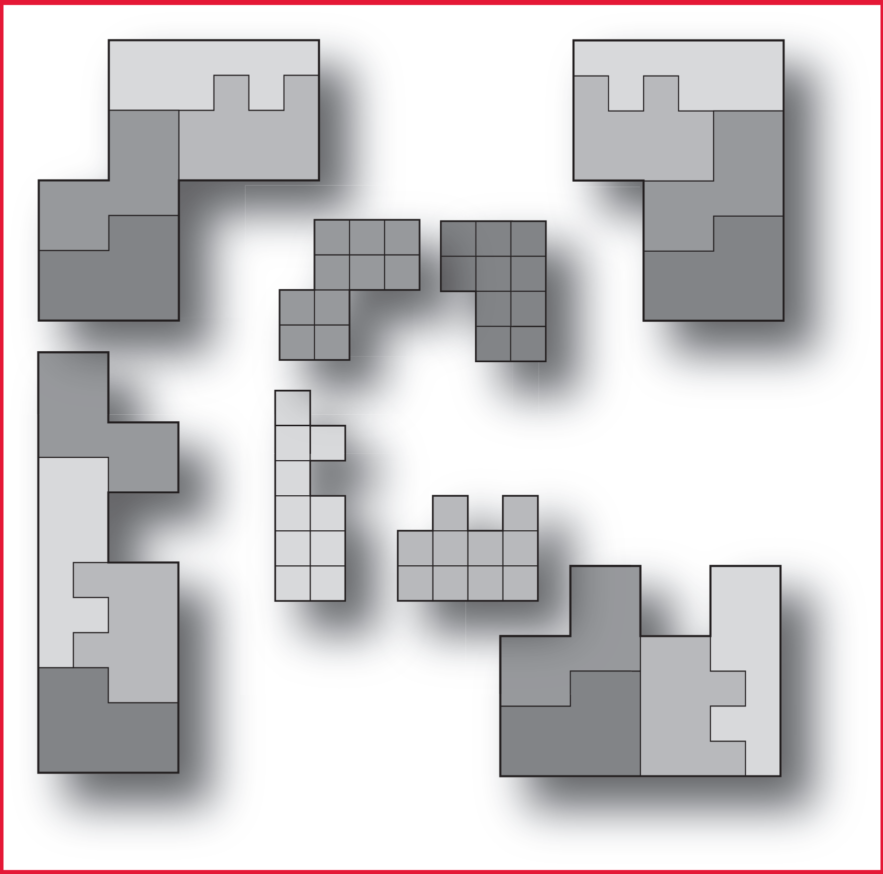

Four figures tile copies of themselves (p. 100)

- Avoiding arithmetic progressions
- Geometry of cubic polynomials in the complex plane
- Sphere packing, magic tricks, Pascal's triangle

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More On Self-Tiling Tile Sets

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Form is not emptiness;
emptiness is not form.

In an earlier article in this MAGAZINE [**7**], I introduced the idea of a *self-tiling tile set*, or hereinafter, *setiset*. A setiset of order n is a set of n planar shapes, each of which can be tiled with smaller replicas of the complete set of n shapes. It is required that the scaling factor be the same for each piece. FIGURE 1 shows an example of order 4 using distinctly shaped pieces. In fact, the definition given in the previous article demanded that the n pieces be distinct, whereas here no such requirement is made. This less stringent definition means that a setiset may be composed of identical pieces, in which case it becomes nothing less than a *rep-tile*, an animal that will figure prominently below.

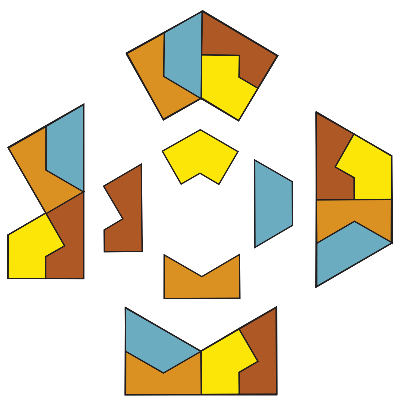

Figure 1 A setiset of order 4.

Since the pieces in any order-n setiset are all paved by the same n shapes, their areas are equal. And since the increase in scale from single piece to compound copy entails an n-fold increase in area, the linear scale factor involved is $\sqrt{n}$. Hence, in the case of pieces with integral side-lengths, $\sqrt{n}$ must be an integer. A setiset of integer-sided pieces, such as polyominoes, using a non-square number of tiles is thus impossible, so that after $n = 4$, the next possibility becomes $n = 9$.

Note also that the shapes employed do not have to be *connected* regions. Disjoint pieces composed of two or more separated islands are equally permissible, if aesthetically less appealing, although our human predeliction for connected pieces is difficult to defend on mathematical grounds. Setisets that include such pieces may thus be described as *disconnected* or *weakly-connected* (when piece islands join only at a point),

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while specimens composed of connected pieces that are also distinct in shape we shall call *perfect*.

How are setisets produced? In the absence of any known alternative, in the previous article I described a computer program that searched for solutions using polyominoes of a given size. Practical considerations meant that its trawls were restricted to sets of four pieces only, using polyominoes of sizes up to 8. These were major limitations.

Since then, however, a fresh insight has revealed a way to produce such sets using almost any (non-prime) number of pieces, and in a far greater variety of shapes. Moreover, the method is simple to the point of child's play and requires no computer. My main purpose here is thus to describe this technique, as well as to present a selection of the many new setisets it has brought to light.

Extracting setisets from rep-tiles

A *rep-tile* of order n (or *rep-n*) is a planar shape that can be tiled with n smaller duplicates of itself. It is easily seen that all triangles and parallelograms are rep-tiles, whereas a specimen of any order, n, may be created simply by stacking n rectangles of size $1 \times \sqrt{n}^{\,-1}$ on top of each other. The size of the resulting pile is then $\sqrt{n} \times 1$, which has the same shape as its components, the property required, albeit that the resulting figure is less than exciting. Skipping rep-tiles of orders 2 and 3, which are similarly dull, FIGURE 2 presents some more attractive examples of order 4, labeled R1 to R12. This list of examples is not exhaustive, but includes most of the interesting specimens I could discover in the literature and on the web [**4, 3, 8, 2, 1, 5, 6**].

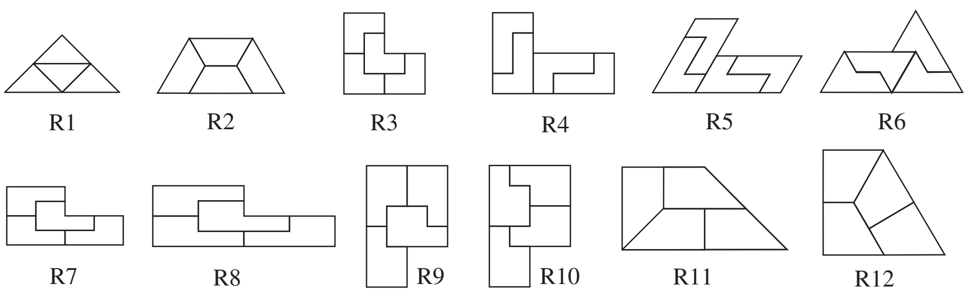

Figure 2 Some rep-tiles of order 4.

Consider rep-tile R6 in FIGURE 2, the hoary "sphinx." It is of order 4. As a rep-tile, we think of it as a single shape divided into units. However, considered as a collection of separate pieces, these units also form a setiset—one in which the four pieces are of the same shape.

Seen thus, we might wonder: Is there some way in which the identical pieces in this setiset can be reshaped so as to render them distinct, but without losing their self-replicating property?

There is. In fact, there is more than one way. The trick is to introduce a second copy of the same setiset. By combining these two sets in different ways, we can form distinct shapes that preserve their self-replicating property. In practice, what this boils down to is dissecting the rep-tiles appropriately.

FIGURE 3 shows three copies of the sphinx, each with its four sphinxlets. Suppose now that the rep-tile is dissected into two smaller pieces of equal area, each composed of a pair of sphinxlets. The result is a sphinx partitioned into two parts, such as *A* and B, having different shapes. Taking next a second copy of the same rep-tile and repeating the process, but now using a different cut along the edges of the sphinxlets, we will then have four pieces, all different in shape. A further repeat will exhaust all possibilities, to leave us with a total of six distinct pieces, all shown in FIGURE 3.

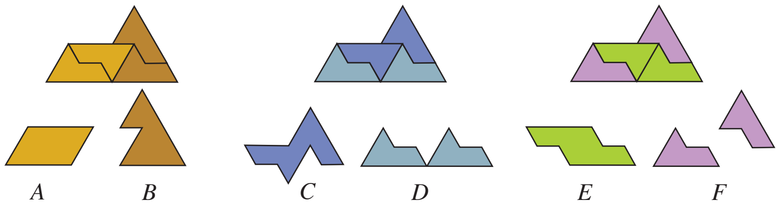

Figure 3 Three ways to dissect the sphinx.

Four of the resulting pieces are connected, A, B, C, E, one weakly-connected, D, and one disjoint, F. Now the pair A and B will together tile the bigger sphinx. The same goes for C and D. So the four pieces A, B, C, and D together tile two separate sphinxes. But two sphinxes are exactly the units from which A, B, C, and D are constructed. So, if these four pieces can pave two sphinxes, and two sphinxes are all we need to create A or B or C or D, then $\{A, B, C, D\}$ must be a self-tiling set. The conclusion is confirmed in FIGURE 4.

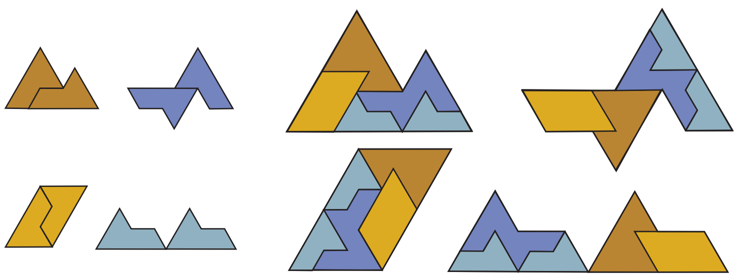

Figure 4 A setiset derived from the sphinx.

The four pieces are each composed of 2×6 unit triangles, and are thus dodeciamonds. To assist the eye, the outlines of the component sphinxlets in the smaller pieces have been left visible. See how the orientation of the two sphinxes making up each small piece is exactly mirrored by that of the larger (two-tone) sphinxes that compose their compound copies.

Above, I said there is more than one way to achieve the goal. This is because $\{A, B, C, D\}$ is not the only set of four we could have chosen. Beside the pairs A, B and C, D, the pair E, F also tiles the sphinx. Hence $\{A, B, E, F\}$ and $\{C, D, E, F\}$ are also setisets, although piece F may be thought less attractive because it is disconnected. Similarly, it is perhaps regrettable that one of the pieces, D, in our first setiset is itself only weakly connected, but the rep-tile structure affords us no better alternative.

Happily, however, this is but the start of our exploration. The rep-tile dissection technique just described is applicable to any rep-tile of order 4, and there remain several more specimens in FIGURE 2 to examine.

Not every rep-4 rep-tile can be dissected into six distinct pairs as above. Readers can check for themselves that rep-tiles R1, R2, R3, R4, and R5 in FIGURE 2 yield at most five different pair-shapes. Four is sufficient for them to form a setiset, but they all include weakly-connected or disjoint pieces.

Turning next to rep-tile R7, a hexomino, inspection reveals it as the first to admit of dissections resulting in four fully connected pieces, to be seen in the setiset of FIGURE 5. Again, the orientation of the two hexominoes that make up each smaller piece are repeated in the compound constructions. This is a characteristic feature of every setiset of order 4 derived via the rep-tile dissection method.

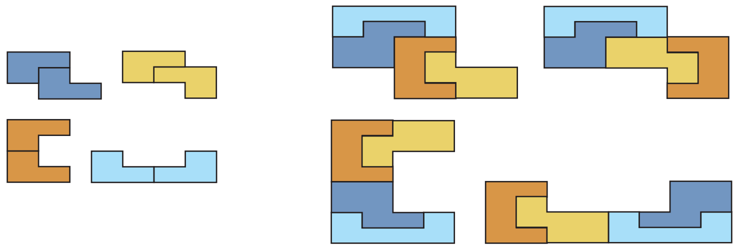

Figure 5 A setiset based on rep-tile *R7* in Figure 2.

As might be expected, the close resemblance between rep-tiles R7 and R8 in FIGURE 2 is echoed in the setisets they give rise to, the one resulting from R8 being an exact analog of the one shown in FIGURE 5. Surprising perhaps is that R3, which again resembles the structure of both R7 and R8, fails even to yield four distinct connected pieces.

A more interesting case is rep-tile R9, which has the shape of a P-pentomino. FIGURE 6 shows the various smaller rep-tile pairs into which it can be dissected, with again one non-connected piece (F).

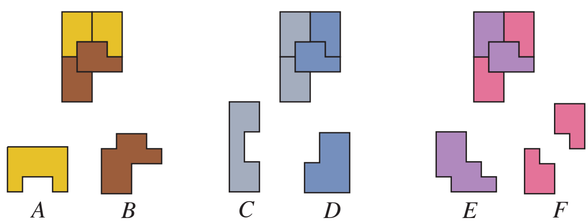

Figure 6 One of the two *P*-shaped rep-tiles dissected thrice.

Leaving aside F (along with its complement, E), there remain four fully-connected pieces that together pave the pentomino twice, which is all that is required of them to create a setiset. It can be seen in FIGURE 8, at left.

There exists, however, a second version of this same rep-tile in which the paving is very slightly different; see R10 in FIGURE 2. The difference between the two versions is so insignificant as to be routinely overlooked. Nevertheless, even such a tiny difference makes for the distinct set of pairwise dissections seen in FIGURE 7, as comparison with FIGURE 6 will show.

Here again, aside from piece Z, there remain four connected shapes that can pave two P-pentominoes. The associated setiset can be seen at right in FIGURE 8, alongside that using pieces from FIGURE 6. However, bearing in mind that a setiset results from any four pieces able to tile the rep-tile twice, there remain yet two more connected-piece setisets to be extracted here. They are those that arise by taking one pair from

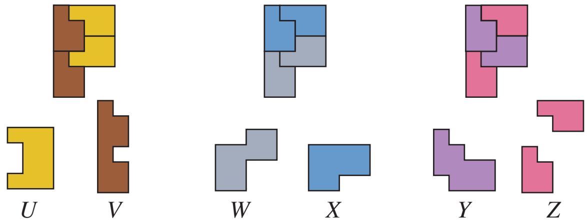

Figure 7 The alternative *P*-shaped rep-tile dissected thrice.

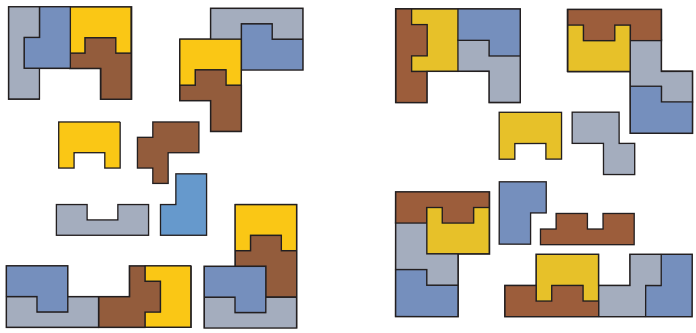

Figure 8 The setisets derived from Figure 6 (left) and Figure 7 (right).

FIGURE 6 together with one pair from FIGURE 7. The resulting two new solutions based upon $\{A, B\} \cup \{W, X\}$ and $\{C, D\} \cup \{U, V\}$ are pictured in FIGURE 9.

Of course, $\{A, B\} \cup \{U, V\}$ and $\{C, D\} \cup \{W, X\}$ also form setisets, but contain two pieces that are alike: A and U in the first, D and X in the second. Even so, the P-pentomino rep-tile proves itself a remarkably rich source of setisets.

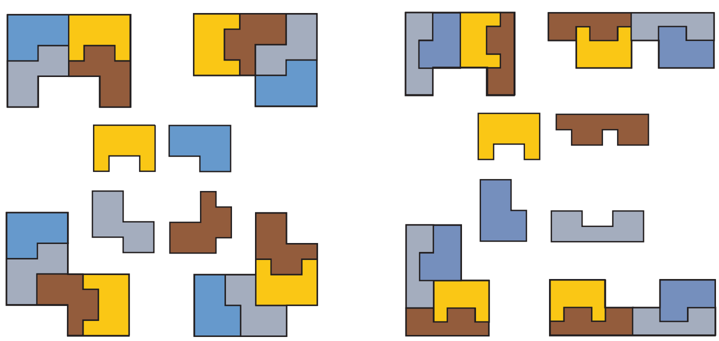

Figure 9 The pieces in Figures 6 and 7 yield two further setisets.

The rep-tile R11 in FIGURE 2 is a polyform built up from units having non-integral side-lengths: 1, 1, 2, and $\sqrt{2}$. It can be dissected into two connected halves in two different ways to result in the setiset of FIGURE 10.

This brings us to the final example in FIGURE 2, rep-tile R12, the shape of which is one quarter (quadrant?) of a regular hexagon. Once again, decomposing the rep-tile into two connected halves in two different ways results in a further setiset of order 4. It is the one with which we started, pictured in FIGURE 1.

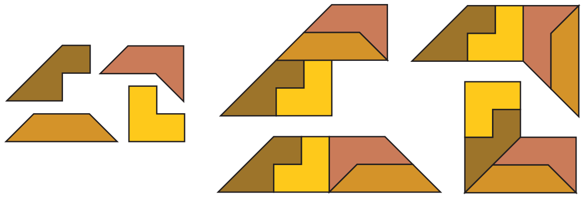

Figure 10 A setiset with non-integral side length pieces.

Higher-order setisets

The dissection method can be applied to rep-tiles beyond order 4, but not in every case. This follows from the fact that the pieces in any setiset are equal in area. Thus, if a set is derived from a rep-tile (not all are), the number of rep-tile units contained in each piece must be the same (and also be greater than 1). Hence, n, the order of the rep-tile, must be divisible by an integer, showing that prime orders will not work. Also, as the number of pieces goes up, so do the number of dissections required. Hence, if d is the number of times the rep-tile is to be dissected, and p is the number of parts into which it is split, then $d \times p = n$. A first step in constructing a setiset of order n is therefore to look at the factors of n so as to see how many dissections of what kind will be needed.

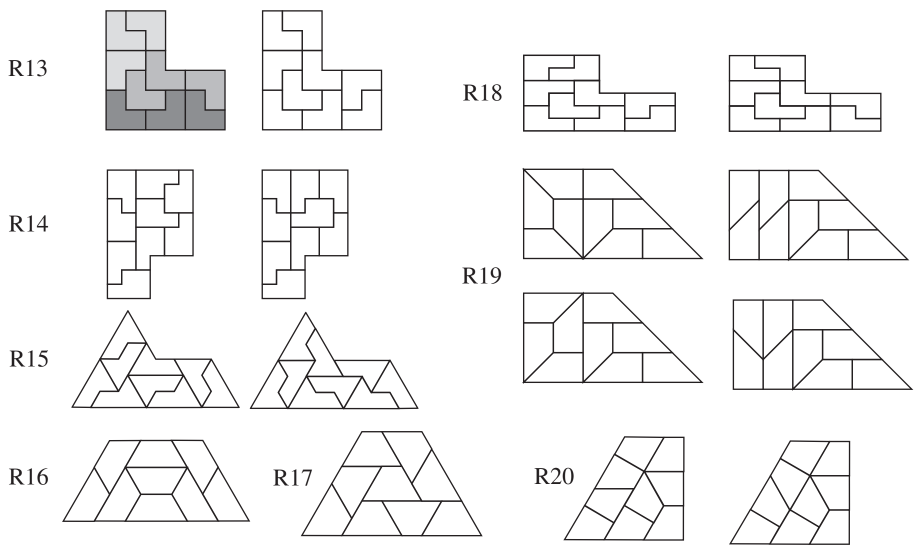

Figure 11 Some rep-tiles of order 9.

FIGURE 11 shows some rep-tiles of order 9, labeled R13–R20. Note the return of shapes earlier seen in FIGURE 2, in fact all except R17. These are among other rep-tiles that can be inflated to form new specimens of higher order. Several of these examples admit more than one paving. FIGURE 12 shows a setiset of 9 pieces (above) derived from rep-tile R13. Being of order $9 = 3 \times 3$, the only possible values for d and p are $d = 3$ and $p = 3$, indicating a need for three dissections into three pieces. The piece shapes seen were arrived at by trial and error. The dissections of the rep-tile that give rise to them can be seen in the L-shapes at the bottom of FIGURE 12. Deciding on how to make these dissections may look like a difficult task, but is really no more than a trivial puzzle. The requirement here was to thrice cut the rep-tile into thirds without producing any repeated shapes. For example, the shading of R13 at left in FIGURE 11 shows an alternative dissection into three pieces. Had we started with this, the next question would be: Can two further dissections into thirds be found so as to yield a total of nine distinct pieces? For rep-tiles of small order such as this, trial and error suffices to find out. For much larger rep-tiles, a computer program could be written to do the job. In fact, here the goal of nine distinct pieces cannot be achieved, forcing a retry with a new dissection. Even now, both pavings of R13 in FIGURE 11 are needed to attain nine distinct pieces.

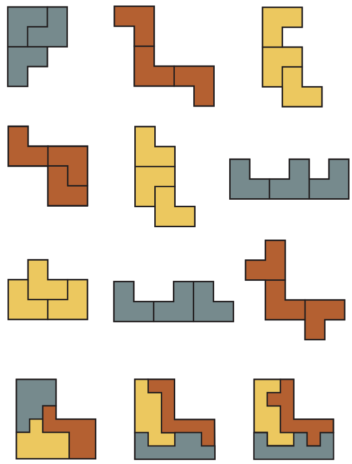

Figure 12 A setiset of order 9.

Returning to FIGURE 12, rather than show the nine pavings of each piece, which would make for an over-intricate picture, the self-tiling property of the set is made manifest more simply. As shown by the thin black lines, every one of the nine pieces can be tiled by three identical L-shapes. Below is seen that these same L-shapes are paved by the complete set of nine pieces. Hence, the latter form a setiset, QED. Note that the three compound L-shapes can tile any piece shape in 3! different ways.

FIGURE 13 shows two comparable perfect setisets derived from rep-tiles R18 and R19 in FIGURE 11. The exotic piece shapes might suggest wizardry to the uninitiated, whereas the method of patiently dissecting the rep-tiles into nine non-congruent regions of three units remains as simple as described above.

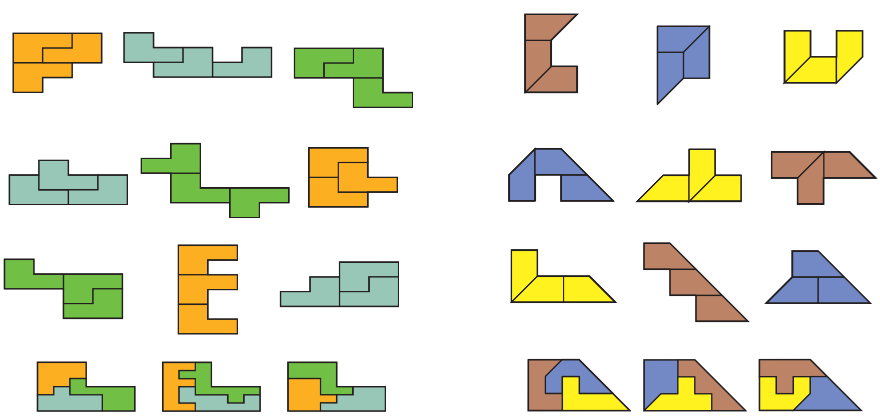

Figure 13 Order 9 setisets based on rep-tiles $R18$ (left) and $R19$ (right).

Not every rep-tile can be dissected so as to yield nine connected pieces. R15, R16, and R17 fall into this category, even though R15, the order-9 sphinx, comes in two versions that make for distinct dissections. It is important to realize that any distinct pavings of a single rep-tile may all be used in deriving the same setiset. As was just seen with FIGURE 12, the extra flexibility thus conferred can be crucial to success. Note that the second paving of R13 (right) results simply by flipping its component 2×3 rectangle at bottom right. This trick of flipping bilaterally symmetric regions can spawn many variant tilings. Four variants among others are shown for R19 in FIGURE 11.

Above, we saw that the rep-4 P-pentomino rep-tile gave rise to four different setisets showing connected pieces. FIGURE 14 shows how the two variants of R14, which is the order-9 version of the same rep-tile, can be dissected into three non-congruent thirds in a total of 11 different ways. Any three of these dissections will form a perfect setiset, provided they include nine distinct pieces. Analysis reveals 130 such cases, a single example of which gives rise to FIGURE 15 left. FIGURE 15 right shows a final example of order 9 based upon R20 in FIGURE 11. By now, the reader should be able to take in the self-tiling property at a glance.

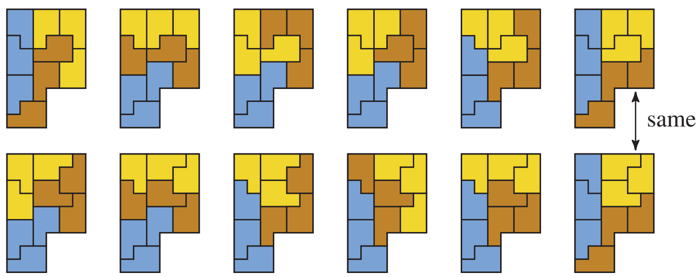

Figure 14 The two variants of rep-tile *R*14 can be dissected in 11 different ways.

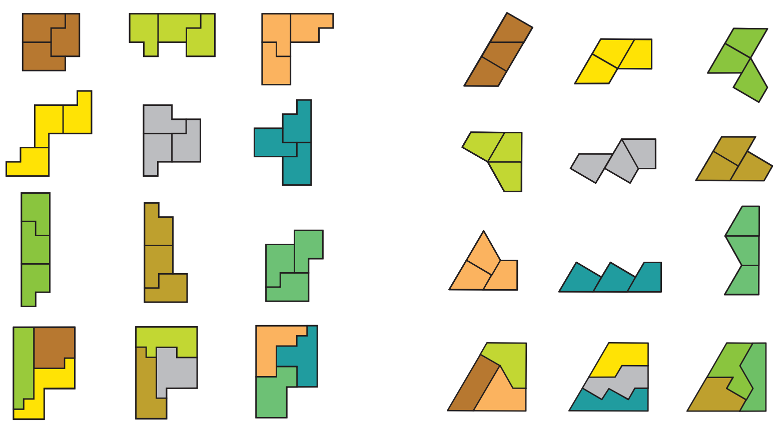

Figure 15 Setisets derived from rep-tiles *R*14 (left) and *R*20 (right).

So much for the unproblematic case of order 9. Similarly, order 16 presents no difficulties. However, a serious obstacle met with when it comes to the non-square, non-prime orders 6, 8, 12 and 14, is the absence of suitable rep-tiles. For example, a search of the literature reveals but one rep-tile of order 6, which is the one formed by a stack of six $1 \times \sqrt{6}$ rectangles. However, since $6 = 2 \times 3$, either $d = 2$ and $p = 3$, or $d = 3$ and $p = 2$, meaning two dissections into three thirds or three dissections into two halves, both of which are impossible to realize. Happily, in this case further thought turned up what I believe to be a previously unknown L-shaped rep-tile of order 6. It can be seen at left in FIGURE 16. In the center is an order-6 setiset it gives rise to, at right of which are seen the two distinct pavings of the rep-tile required. Alas, although alternatives exist, at least one disjoint piece is unavoidable.

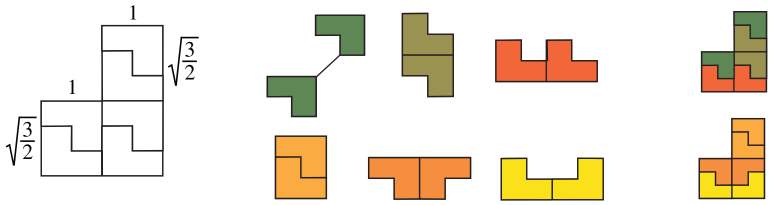

Figure 16 A (new?) order 6 rep-tile and associated setiset that includes a disjoint piece.

Likewise, the only setiset of order 8 yet found is one including four disconnected pieces that is based on a $\sqrt{8} \times 4$ rectangular rep-tile. Excessive as so many disjoint pieces may seem, this does at least furnish a solution, whereas when it comes to orders 12 and 14, solutions are lacking completely.

FIGURE 17 shows an unusual-looking setiset of order 10. It is derived from the rep-tile shown beneath, a right-angled triangle with sides of length 1, 3, and $\sqrt{10}$. Here the factors of 10 dictate five dissections into two halves. The alternative of two dissections into five fifths proves impossible. At left, an example shows how the ten pieces tile one of their own number. If it seems that some pieces are merely flipped versions of others, don't be deceived; all are distinct.

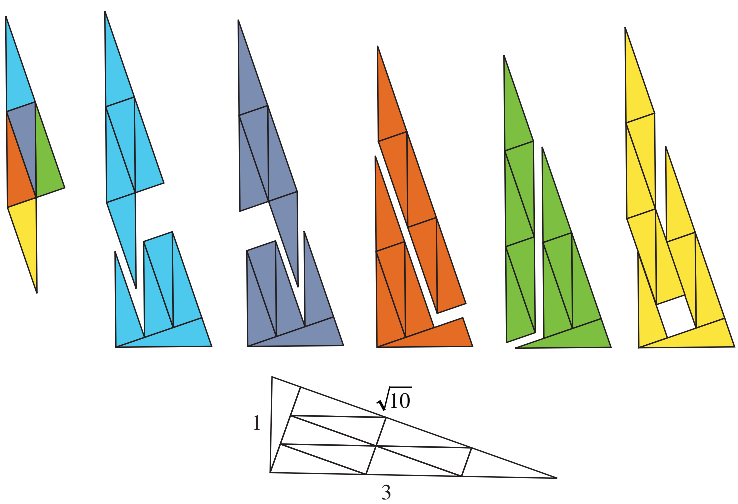

Figure 17 A setiset of order 10 based on the rep-tile shown below.

I conclude this brief tour of the setiset flora with two final examples of order 16. From $16 = 4 \times 4 = 2 \times 8 = 8 \times 2$, we find there are three possibilities. FIGURE 18 makes use of the L-shaped rep-tile already encountered in order-4 and order-9 form, but now of order 16. Four-fold dissections of the L into four non-congruent quarters result in 16 distinct pieces that tile themselves. Finding a rep-tile that can be dissected twice into eight distinct eighths, or eight times into two distinct halves are challenges I leave to the reader.

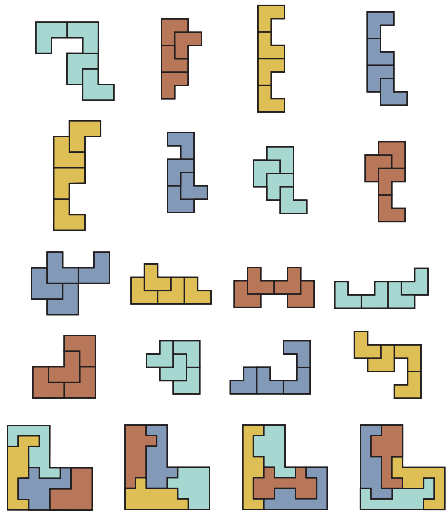

Figure 18 A setiset of order 16.

A search for the setiset in FIGURE 19 was undertaken after noticing that FIGURE 18 resembles a 4×4 geomagic square. For that is exactly what FIGURE 19 is: The four pieces in each row and column (but not the diagonals) all pave a 4×8 rectangular target. Moreover, each of the 16 pieces is itself composed of four 2×4 rectangles. An enlarged copy of any piece can thus be assembled using either the four row or the four column targets, which are of similar shape.

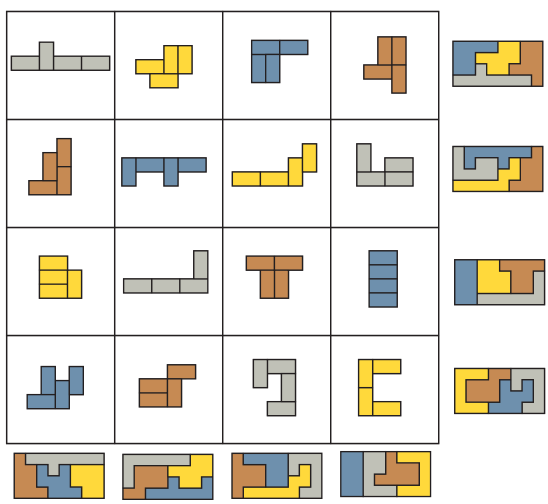

Figure 19 An order 16 setiset arranged so as to form a 4×4 geomagic square.

Double-rep-tiles and self-constructors

Rep-tiles are not the only tile sets to exhibit self-similarity. Besides rep-tiles, there are sets that include more than one kind of shape. For example, a set of two distinct shapes, each of which can be tiled by two smaller duplicates of both, is what I call a *double*-rep-tile.

FIGURE 20 shows an example in which the areas of distinct color are pentiamonds of two kinds: A-shaped (orange) or B-shaped (yellow). Here, the double-rep-tile is composed of the two twice-sized or "big-pentiamonds," A and B, seen at top. Both A and B are each tiled by two A-shaped and two B-shaped pentiamonds. The two tints of orange help to distinguish distinct components. As with rep-tiles, the self-similar properties of double-rep-tiles yield setisets.

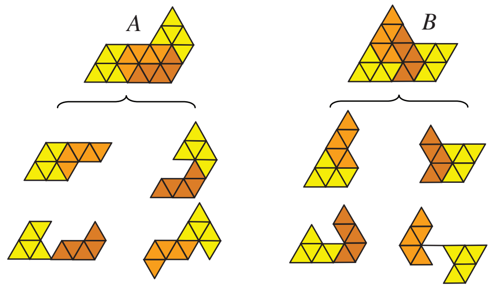

Figure 20 A double-rep-tile: two shapes tiled by two smaller duplicates of both.

Below A and B are seen their two possible dissections into two halves each composed of an A- and a B-pentiamond. They include a weakly-connected and disjoint piece. Since an A-pair will unite to form A, and a B-pair to form B, any two such pairs yield a setiset of order 4. Among the four possibilities, the two pairs immediately under A and B furnish one using connected pieces, shown in FIGURE 21 at left.

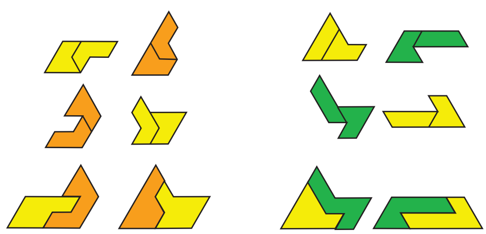

Figure 21 Two setisets derived from two double-rep-tiles.

Unlike setisets derived from rep-tiles, in which pieces are composed of identical units, here they are built up from distinctly-shaped pairs. At the right of FIGURE 21 is a setiset derived from a second double-rep-tile, again composed of two big-pentiamonds.

These double-rep-tiles were discovered with the aid of a computer program that identifies every possible tiling of a big-pentiamond by four pentiamonds. A similar program was used to find a pair of double-rep-tiles based upon polyominoes. Both use P-shaped and V-shaped big-pentominoes. The P-shaped one can be tiled in two different ways, as FIGURE 22 shows.

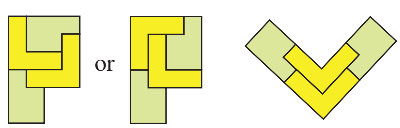

Figure 22 A double rep-tile using P and V pentominoes.

The dissections of these pieces result in three setisets having connected pieces. FIGURE 23 shows that the three share two pieces in common, while two of them differ in one piece only.

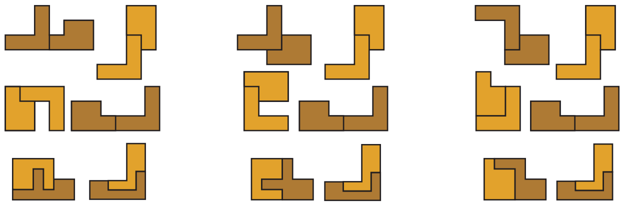

Figure 23 Three order 4 setisets derived from the double-rep-tile of Figure 22.

As before, every piece is composed of two distinct units. After looking at these specimens, it is not difficult to imagine setisets whose pieces are composed of three or more distinct units, a contingency that invites us to contemplate *multi-rep-tiles*. On the other hand are piece sets in which the number of copies of each component shape is not the same in each case. For example, FIGURE 24 shows a remarkable set of three distinct big-pentiamonds labeled A, B, and C. If we let a, b, and c stand for their corresponding pentiamonds, and write $P, Q \leftarrow r, s$ to indicate that both P and Q are tiled by r as well as by s, then the properties of A, B, and C as seen in FIGURE 24 are as follows: $A, B, C \leftarrow 2a + b + c, a + 2b + c, a + b + 2c$, where the 2's indicate two copies.

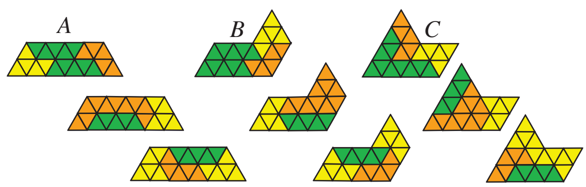

Figure 24 A remarkable set of self-tiling pentiamonds.

No setisets result from dissecting these pieces, but instead can be found a pair of what I tentatively dub *self-constructors* of order 6. FIGURE 25 shows one example. Each of the six pieces in the set is paved by some four of them.

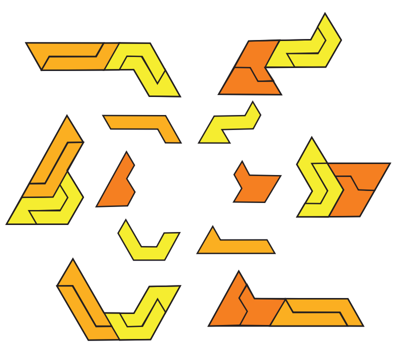

Figure 25 A "self-constructor." Each piece is paved by some four of them.

Note that the three shapes in FIGURE 24 do not qualify as a "triple-rep-tile," which would be a set of three distinct shapes, each of which is tiled by three smaller copies of themselves. I have no idea whether such creatures even exist, although there seems no reason to exclude the possibility. But what FIGURE 24 does demonstrate is that besides multi-rep-tiles of higher degree, there may exist still stranger animals sporting pieces composed of duplicates that occur *in differing numbers*. As to what peculiar forms of self-replicator these may yet lead in the future, who will say?

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Summary This is an extension of the author's previous work on self-tiling tile sets. Hitherto, the only known method for discovering the latter was by means of a brute-force computer search. A new pencil and paper method for extracting such sets from rep-tiles is introduced, and a plethora of new examples presented.

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