Alphamagic Squares

Adventures with turtle shell and yew between the mountains of mathematics and the lowlands of logology.

by Lee C. F. Sallows

"Eleven + two = twelve + one" — Martin Gardner

The following article made its debut as a talk given this summer at the unique Eugene Strens Memorial Conference on Intuitive and Recreational Mathematics and Its History, held at the University of Calgary, Alberta. The conference was designed to mark the University library’s acquisition of the Strens Collection.

Strens, whose home was at Breda in the south of Holland, devoted most of his life to collecting. On his death he left behind him what is probably the world’s most remarkable assemblage of books on recreational math topics.

The author, who lives in the Netherlands and “stumbled across this Aladdin’s Cave while exploring the Dutch mountains,” contacted Martin Gardner about it. Together, they were instrumental in putting the Strens family in contact with Richard Guy of the Department of Mathematics at the University of Calgary, which eventually led to the collection being moved to its new home.

The history of magic squares is a venerable one, reaching back into the legendary past of ancient China. So it is that the simplest, oldest, and most famous square of all, the so-called Lo shu (shu meaning writing, document), is said to have first been revealed on the shell of a sacred turtle which appeared to the mythical Emperor Yu from the waters of the Lo river in the 23rd century B.C. (This is discussed in Camman’s

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The roots of tree worship can be traced down into the rich subsoil of Celtic antiquity, but its true origin is to be sought earlier still, in the reaction of a prehistoric culture steeped in animism and magic to the newly acquired control over fire. The benefits conferred by fire blazing upon the primitive hearth were manifold indeed; an unmitigated blessing as long as plentiful supplies of fallen bough were at hand with which to feed the flames. But the gradual depletion of these reserves would create a dilemma as men turned their thoughts—and their stone axes—to the growing glade and blowing greenwood. For the sources of living timber were protected by taboo; branch and bole rendered inviolable by a system of totemism which saw in every tree the abode of supernatural—possibly vindictive—spirits. Bold would be he who dared incur the vengeance of the tribal totem. But braver still were those who would endure the long, cold, lightless nights of winter! Thus would ritual, sacrifice and magic be pressed into the service of propitiating these arboreal powers.
ramified, not to say widely and haphazardly dispersed. It is clear that the spell cast by the elegant symmetries reflected in these interlocking number patterns has held countless devotees in thrall, eminent mathematician and lowliest layman alike. Hardly a turtle shell has been left unturned in exploring variations on the central theme, so that articles and even books abound devoted to special categories of squares, as well as magic triangles, rectangles, circles, stars, antimagic squares, prime-number squares, multiplicatively magic squares, magic cubes, N-dimensional arrays, and so on. Not least in adding spice to the subject is the variety of simply stated yet peculiarly intractable mathematical problems they give rise to. Nobody knows, for instance, how many distinct consecutive-number specimens there are for any square larger than $5 \times 5$. Barely credible, but true!

A new development of unexpected relevance to this topic is the recovery during 1985 of a unique book, bringing to light an extraordinary parallel between an episode in the reign of King Mi, a historically dubious late-fifth-century tribal chieftain of North Britain, and the Chinese legend of the Lo shu. Apparently misplaced in or about 1888, this book, The Origin of Tree Worship, a privately printed nineteenth-century work of scholarship devoted to a study of Druidical practices and the spread of the yew cult among Celtic and Germanic peoples in pre-Christian Europe, recently surfaced again during a reorganization of bookshelves at the British Library (formerly the British Museum) in London.

Mysteriously abandoned after preliminary publication in a sparse edition of just six sample copies, the rediscovered volume is in all probability the only surviving exemplar (see photo), and its reappearance after nearly one hundred years has caused a considerable ripple in philological circles. The reason for this lies in a wealth of unmistakable internal evidence showing that the author must have been borrowing from medieval manuscript material previously believed lost in the fire that destroyed so much of the famous Cottonian collection of priceless early English documents, while it was housed at Little Deans Yard, Westminster, in 1731. As such, The Origin of Tree Worship is presently the subject of minute scrutiny by experts and, quite apart from the urgent questions thrown up by the provenance of its cited material, is already shedding light in several areas of paleographical research. Readers interested in further details (including a review of conflicting evidence as to the real identity of its author) may care to consult the British Library Department of Occidental Manuscripts Internal Report No. 2704/1729, as well as the forthcoming article by J. Allardyce and M. Sandeford, scheduled to appear in the Journal of English and Germanic Philology.

Returning to our present purpose: among other previously unrecorded Celtic myths alluded to in The Origin of Tree Worship is an account of a pilgrimage made by King Mi to a sacred grove in Eohdali, Valley of the Yews, where, following pious observance of symbolical pagan rites, a runic charm or magical formula is revealed to him, scored on the bole of the hallowed Li, eldest of yews. Runes, it will be recalled, are thin angular characters suited to incision on wood, stone, metal, and so forth; their employment by primitive (chiefly Scandinavian) tribes was seldom for practical purposes of communication, but almost always bore magico-ritualistic significance. An excellent survey of the subject is Runes: An Introduction by R. W. V. Elliott (Manchester University Press, 1980).
As an amateur runologist fortunate enough to have been granted a privileged view of this exciting find (a facsimile edition is presently in preparation), I was naturally drawn to deciphering the runic charm reproduced in the book along with the narrative of King Mi (Figure 3). It is a mark of the great advances made in paleography over the intervening years that a problem that seems to have baffled solution in 1887 (the date of publication) offers little difficulty to the modern investigator. In Figure 4, modern usage replaces Old English orthography. At first I was much puzzled by the pattern of cardinal number-names thus disclosed, and it was only on writing them out in more perspicuous form that understanding eventually dawned (see Figure 5). As the reader can easily verify, the sum of the three elements occurring in every row, column, and diagonal is the same: 45. What we have here, in other words, is a familiar $3 \times 3$ magic square.

Fascinating as this parallel with the Lo Shu legend is, however, it remains worth noting that although distinct, the nine numbers appearing in the runic square fail to form a consecutive series, as in their Chinese counterpart. Nevertheless, the Li Shu (as I suppose it can hardly otherwise be called) bears closer examination. Seeking for something to warrant a supernatural manifestation on a sacred yew tree, and having already been prompted through registering a small coincidence while transliterating the runes, I soon discovered that the number of runes—and, thus, by chance, the number of modern English letters—making up the three words used in every row, column, and diagonal is also identical: there are twenty-one! (The coincidence between modern and archaic word lengths will seem of less moment to readers familiar with the normal course of etymological development from Old English forms: two = twa, five = fife, eight = eahte, twelve = tuolfe, and so on).

Moreover (and here I began to appreciate the potency of this singular thaumaturgical device), writing out the rune or letter totals associated with each number-name not only results in a second magic square, the numbers now emerging do indeed comprise an unbroken consecutive series (Figure 6). Furthermore, since no English cardinal number-name, old or new, is shorter than three let-
Worship in hope of finding further details. Alas, nothing of interest is to be found there, save a bare record of the legend quoted as evidence of yew practices in Northumbria at that period, together with a conjecture that the formula had probably been credited with healing powers and would have been worn on talismans to ward off evil. Nor has any external enquiry succeeded in eliciting further amplification. The Li shu, it appears, exists as a unique, isolated prototype, and any subsequent developments it may have given rise to have long since been lost to us, buried in the dust of history.

Obscure as its origins remain, clearly the rediscovery of this fantastic formula immediately provokes a host of tantalizing questions and contingencies quite independent of the historical, mythological, philological, and, indeed, criminological issues raised in connection with The Origin of Tree Worship itself. In fact, as the following will show, the Li shu furnishes a point of departure into an exciting new genre, a hitherto undreamed-of field, perhaps best described as a kind of recreational department of Computational Linguistics. I refer to the exploration of alphamagic squares.

3 x 3 Alphamagic Squares

Alphamagic is the word I use to describe any magic array (whether square, rectangular, triangular, N-dimensional, etc.) that remains magic when all of its entries are replaced by numbers representing the word length, in letters, of their conventional written names (thus, one becomes 3). Plainly, a square that is alphamagic in one language need not be so in another (and nonalphabetic languages are irrelevant in this context). It will be convenient to refer to the letter count of a number-word as the logorithm—or log, for short—of the original number (logos = word, arithmos = number). Logarithm should not be confused with a similar word coined by a Scotchman called Napier in 1614. Where unstipulated, “natural” logs or \( \log_{\text{english}} \) will be assumed; hence \( \log 15 + \log 3 = \log_{\text{french}} 69 \) since \( 7 + 5 = 12 \), the number of letters in soixante-neuf.

By magic I shall mean any arrangement producing a constant sum along its various orthogonals and diagonals, regardless of whether the elements involved are distinct or not. Naturally, a square showing repeated entries is less interesting than one in which all are different. The order of a square refers to its size—the num-

---

**Letter Counts from Figure 4**

<table>
<thead>
<tr>
<th>4</th>
<th>9</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>

---

**General Formula**

<table>
<thead>
<tr>
<th>(a+b)</th>
<th>(a-b+c)</th>
<th>(a+c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a-b+c)</td>
<td>(a)</td>
<td>(a+b-c)</td>
</tr>
<tr>
<td>(a-c)</td>
<td>(a+b+c)</td>
<td>(a-b)</td>
</tr>
</tbody>
</table>

---

**A Table of Natural Logs**

<table>
<thead>
<tr>
<th>Numbers</th>
<th>Logarithms</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 ...</td>
<td></td>
</tr>
<tr>
<td>4 3 3 5 4 4 3 5 5 4 3 6 6 8 8 7 9 8 8 6 9 9 11 10 10 9 11 11 10 6 9 9 11 10 10 ...</td>
<td></td>
</tr>
</tbody>
</table>

---

**Figure 8. The natural numbers 0 through 35 together with their logarithms. Connecting lines indicate triples in which both numbers and logarithms form regular arithmetic series.**
number of cells on a side. The Li shu is thus an English alphamagic square of order 3, having as additional properties that its logarithms are distinct, consecutive, and minimal (that is, they comprise a set of the smallest possible non-aliike logarithms existent in English). Order-1 and order-2 squares need not detain us, as a moment’s consideration will show. With these few conventions established, we are ready to pursue the main theme.

As a tentative entry into the unfamiliar terrain, it is natural to wonder if there are any 3×3 alphamagic squares other than the one produced above. Useful in this connection is the general formula for order 3 shown in Figure 7 (and due to Édouard Lucas), since both numbers and logarithms in an alphamagic square must satisfy the relations it exemplifies.

Note that the three elements on each straight-line bisector through the center form a set of four 3-term arithmetic series (that is, they show a constant difference between adjacent terms: \(a-b+c=a -\frac{1}{2}(a+b-c)\), for instance). Then one obvious initial step is to search for arithmetic triples whose logarithms share the same property.

Figure 8 lists the cardinal numbers from 0 to 35 together with their English logarithms. Taking for illustration a center number \(C\) of 15, consider in turn arithmetic triples formed by \(C\) and its equidistant neighbors \(C-1\) and \(C+1,\) \(C-2\) and \(C+2,\) and so on. Note down those cases in which \(\log(C-N),\) \(\log C (=7),\) and \(\log(C+N)\) also form arithmetic triples. When \(N = C\) we can go no further, since \(C - N = 0.\) By now we shall have a list of pairs of associated arithmetic triples (Figure 9).

If there are any 3×3 alphamagic squares with a center number of 15 (and we already know there is one), at least four of these five cardinal-number triples must appear in it: one along each straight-line bisector, including diagonal bisectors.

Selecting now the first two triple-pairs on the list for closer scrutiny, write them into the diagonals of corresponding matrices (Figure 10). The choice of diagonals here is not critical; alternative linear cell-groups might be used. We argue that since the latter will have to be occupied by two of the listed cardinal-number triples, testing each pair in turn in these positions will comprise an exhaustive check of all possibilities. Note that changing the order in which a given pair is written into the diagonals merely creates rotations or reflections of the same configuration.

Referring back to the general formula, we find that the magic constant of any square is always 3

<table>
<thead>
<tr>
<th>Cardinal Numbers</th>
<th>English Logarithms</th>
</tr>
</thead>
<tbody>
<tr>
<td>12 15 18</td>
<td>6 7 8</td>
</tr>
<tr>
<td>11 15 19</td>
<td>6 7 8</td>
</tr>
<tr>
<td>8 15 22</td>
<td>5 7 9</td>
</tr>
<tr>
<td>5 15 25</td>
<td>4 7 10</td>
</tr>
<tr>
<td>2 15 28</td>
<td>3 7 11</td>
</tr>
</tbody>
</table>

Figure 9

<table>
<thead>
<tr>
<th>Cardinal Numbers</th>
<th>English Logarithms</th>
</tr>
</thead>
<tbody>
<tr>
<td>12 11</td>
<td>6 6</td>
</tr>
<tr>
<td>15</td>
<td>7</td>
</tr>
<tr>
<td>19 18</td>
<td>8 8</td>
</tr>
</tbody>
</table>

Figure 10

Li Shu

<table>
<thead>
<tr>
<th>Fundamental</th>
<th>Second Harmonic</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 22 18</td>
<td>105 122 118</td>
</tr>
<tr>
<td>28 15 2</td>
<td>128 115 102</td>
</tr>
<tr>
<td>12 8 25</td>
<td>112 108 125</td>
</tr>
</tbody>
</table>

Figure 11
**ALPHA.BAS**

```
10 '******************************************************************************
20 '* Program: ALPHA.BAS (GWBasic)
30 '* Purpose: To generate and print out all 3 x 3 alphamagic squares formable using 9 distinct cardinals in
40 '* the range 0 - NMAX. Index numbers and logarithm
50 '* squares are printed alongside.
60 '* Author: Lee C.F. Sallows
70 '* Date: Guy Fawkes Day (November 5th), 1985
80 '******************************************************************************

90 Array definitions:
100 A Logarithms of 0 - NMAX (loaded as data)
110 B Arithmetic triples showing arithmetic logarithms
120 C Logarithm-triple counterparts to numbers in B
130
140 NMAX=109 'In this example
150 DIM A(NMAX): DIM B(100,3): DIM C(100,3)
160
170 Data is loaded into A; A(n) thus corresponding to log n.
180 Contingent triple CENTRE numbers are considered in turn.
190
200 FOR I=0 TO NMAX: READ A(I): NEXT I
210 FOR CEN = 4 TO NMAX-4
220
230 Starting with the two highest-lowest values possible (BOUND)
240 and working inwards, A-elements equidistant about A(CEN) are
250 checked with A(CEN) to see if they form arithmetic triples. If so,
260 COUNT is incremented, the number-triple is stored in B and its
270 associated log-triple stored in C.
280 Provided at least 4 triple-pairs are found we proceed to the next
290 stage; otherwise reset COUNT and take the next CENTRE number.
300
310 IF CEN(NMAX-CEN) THEN BOUND=CEN ELSE BOUND=(NMAX-CEN)
320 IF A(CEN)<>A(CEN-BOUND) THEN GOTO 380
330 COUNT=COUNT+1
340 B(COUNT,1)=(CEN-BOUND): B(COUNT,2)=CEN: B(COUNT,3)=(CEN+BOUND)
350 C(COUNT,1)=A(CEN-BOUND): C(COUNT,2)=A(CEN): C(COUNT,3)=A(CEN+BOUND)
360 BOUND=BOUND-1: IF BOUND<1 THEN GOTO 340
370 IF COUNT<4 THEN GOTO 960
380
390 B now contains 4 or more arithmetic triples, C their associated
400 logarithms. Using I and J to address every possible pair of
410 B-triples in turn, we deal with them as though written into the
420 diagonals of a 3 x 3 test matrix, thus:
430
440 | B(I,1)  ...  B(J,3) |
450 |     ...  CEN     |
460 | B(J,1)  ...  B(I,3) |
470
480 Magic-fulfilling values are now calculated for the remaining empty
490 cells, checking one at a time to see if their logarithms also
500 satisfy magic conditions in the associated log-matrix.
510
520 FOR I=1 TO COUNT-1
530 FOR J=I+1 TO COUNT
540 CL=3+CEN-B(I,1)-B(J,1) 'CL = centre left column number
550 IF CL>NMAX OR CL<0 THEN GOTO 950 'entries must be within limits
560 IF A(CL)<3*A(CEN)-C(I,1)-C(J,1) THEN GOTO 950 'check left column log.
```

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times its center number. Therefore, if the left-hand matrix is to be magic, the middle cell in its top row will have to contain \((3 \times 15) - (12 + 11) = 22\). Similarly, if the square is to be alphamagic, the corresponding cell in the right-hand matrix will hold \((3 \times 7) - (6 + 6) = 9\). Now, does \(\log 22 = 9\)? Yes, it does. So far so good. Consider next the middle cell, right-hand column. Does \(\log(45-29) = 21-14\)? Again, yes. Fine; next take the bottom row. Does \(\log(45-37) = 21-16\)? Yes! This is too good to last; cross your fingers and try the last vacant cell. Does \(\log(45-31) = 21-14\)? No! Yuck...
Program AlphaMagicSquaresOfOrder3;
Const Range=109;
   Logarithm : Array[0..Range] Of Integer = ( 4,3,3,5,4,4,3,5,5,4,
            3,6,6,8,8,7,7,9,8,8,
            6,9,9,11,10,10,9,11,11,10,
            6,9,9,11,10,10,9,11,11,10,
            5,8,8,10,9,9,8,10,10,9,
            5,8,8,10,9,9,8,10,10,9,
            5,8,8,10,9,9,8,10,10,9,
            7,10,10,12,11,11,10,12,12,11,
            6,9,9,11,10,10,9,11,11,10,
            6,9,9,11,10,10,9,11,11,10,
            10,13,13,15,14,14,13,15,15,14);
Var Square : Array[-1..1, -1..1] Of Integer;
Var center, counter : Integer;

Function Min( x, y : Integer ):Integer;
Begin If x<y Then Min:=x Else Min:=y End;

Procedure ReportAlphaMagicSquare( c,di,d2, t,1,r,b : Integer );

Var i,j : Integer;

Begin
   Square[-1,0]:=l; Square[1,0]:=r; Square[0,-1]:=b; Square[0,1]:=t;
   Square[-1,1]:=c-d2; Square[1,1]:=c+d2;
   Square[-1,-1]:=c-di; Square[1,-1]:=c+d1;
   Square[0,0]:=c;
   Writeln(' alphamagic square No. ',counter);
   For i:=-1 To 1 Do Begin For j:=-1 To 1 Do
      Do Write( Square[j,i]:6 );
      Write(' <-> ');
      For j:=-1 To 1 Do Write( Logarithm[ Square[j,i] ]:6 );
      Writeln(' ')
   End;
   Writeln(' ')
End;{=procedures of AlphaMagicSquaresOfOrder3=}
Procedure MaybeAlphaMagicSquare( cen,dist1,dist2 : Integer );

Var MagicConstant, AlphaMagicConstant,
    left,right,top,bottom : Integer;

Function MagicTriple( x,y : Integer; Var mid : Integer): Boolean:
Begin MagicTriple := False;
    mid := MagicConstant \( -x-y \);
    If (mid>0) And (mid\( \leq \)Range) And (mid \( \neq \) y)
    { the third test eliminates trivial solutions }
    Then MagicTriple := (AlphaMagicConstant =
         Logarithm[ x ]
         + Logarithm[ mid ]
         + Logarithm[ y ] )

End;

Begin
    MagicConstant := 3*cen;
    AlphaMagicConstant := 3*Logarithm[ cen ];
    If MagicTriple( cen-dist1,cen-dist2, left )
    Then If MagicTriple( cen-dist1,cen+dist2, top )
    Then If MagicTriple( cen-dist1,cen+dist2, bottom )
    Then If MagicTriple( cen+dist1,cen+dist2, right )
    Then Begin counter:=counter+1;
        ReportAlphaMagicSquare( cen,dist1,dist2,
                       top,left,right,bottom )
    End
End;

Procedure GenerateSquaresAroundCenter( c : Integer );

Var k,l : Integer;

Function LogoArithmeticTriple( cen,dist : Integer): Boolean;
Begin LogoArithmeticTriple :=
   Logarithm[ cen ] - Logarithm[ cen-dist ] =
   Logarithm[ cen+dist ] - Logarithm[ cen ]
End:

Begin
    For k:=Min( c,Range-c) DownTo 1
    Do If LogoArithmeticTriple( c,k )
        Then For l:=Min( c,Range-c) DownTo k+1
            Do If LogoArithmeticTriple( c,l )
                Then MaybeAlphaMagicSquare( c,k,l )
End;

Procedure Main Program
Begin
    counter:=0;
    For center:=4 To Range-4
    Do GenerateSquaresAroundCenter( center )
End.

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## Alphamagic Squares Nos. 1–10

<table>
<thead>
<tr>
<th>Index Numbers</th>
<th>Alphamagic Squares</th>
<th>Logarithm Squares</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>No. 1</strong> (the <em>Li shu</em>)</td>
<td>five twenty-two eighteen</td>
<td>4 9 8</td>
</tr>
<tr>
<td>5 22 18</td>
<td>twenty-eight fifteen two</td>
<td>11 7 3</td>
</tr>
<tr>
<td>28 15 2</td>
<td>twelve eight twenty-five</td>
<td>6 5 10</td>
</tr>
<tr>
<td>12 8 25</td>
<td>eight nineteen eighteen</td>
<td>5 8 9</td>
</tr>
<tr>
<td>25 15 5</td>
<td>twenty-five fifteen five</td>
<td>10 7 4</td>
</tr>
<tr>
<td>12 11 22</td>
<td>twelve eleven twenty-two</td>
<td>6 6 9</td>
</tr>
</tbody>
</table>

| **No. 3** | fifteen seventy-two forty-eight | 7 10 10 |
| 15 72 48 | seventy-eight forty-five twelve | 12 9 6 |
| 78 45 12 | forty-two eighteen seventy-five | 8 8 11 |
| 42 18 75 | eighteen sixty-nine forty-eight | 8 9 10 |
| 18 69 48 | seventy-five forty-five fifteen | 11 9 7 |
| 75 45 15 | forty-two thirty-one seventy-two | 8 9 10 |
| 42 31 72 | twenty-one sixty-six forty-eight | 9 8 10 |
| 21 66 48 | seventy-two forty-five eighteen | 10 9 8 |
| 72 45 18 | forty-two twenty-four sixty-nine | 8 10 9 |
| 42 24 69 | four one hundred one fifty-seven | 4 13 10 |
| 4 101 57 | one hundred seven fifty-four one | 15 9 3 |
| 107 54 1 | fifty-one seven one hundred four | 8 5 14 |
| 51 7 104 | forty-four sixty-one fifty-seven | 9 8 10 |
| 44 61 57 | sixty-seven fifty-four forty-seven | 10 9 8 |
| 67 54 41 | fifty-one forty-seven sixty-four | 8 10 9 |
| 51 47 64 | five one hundred two fifty-eight | 4 13 10 |
| 5 102 58 | one hundred eight fifty-five two | 15 9 3 |
| 108 55 2 | fifty-two eight one hundred five | 8 5 14 |
| 52 8 105 | forty-five sixty-two fifty-eight | 9 8 10 |
| 45 62 58 | sixty-eight fifty-five forty-eight | 10 9 8 |
| 68 55 42 | fifty-two forty-eight sixty-five | 8 10 9 |
| 52 48 65 | forty-six seventy-eight one hundred one | 8 12 13 |
| 46 78 101 | one hundred thirty seventy-five twenty | 16 11 6 |
| 130 75 20 | forty-nine seventy-two one hundred four | 9 10 14 |

*Figure 12. The first ten English alphamagic squares of order 3, together with their logarithm squares.*
ber. Try it; one doing is worth a hundred seeings (old Northumbrian proverb). But what about all the other possible center numbers? To canvass all cases systematically, we need to begin with $C = 4$ (a lower number would be pointless, at least 4 distinct triples being demanded in any square), considering in turn $C = 5$, $C = 6$, ..., for as long as we wish to pursue the problem. Clearly, if ever a task was made for a computer, this is it.

The algorithm sketched above represents just one possible method, here incorporated into the simple Basic program labeled ALPHABAS; a Pascal form, AlphamagicSquaresOfOrder3, was later prepared by my colleague Victor Eijkhout (see pages 34–37). Once the program was running, I was able to amuse myself over several weeks by exploring the alphamagic realm of order 3. It is a pursuit I can recommend to others. As one proceeds, the impression slowly grows of having ventured into a space offering almost unlimited recreational potential.

Besides the two examples already signalled, are there many other $3 \times 3$ English alphamagic squares? The answer is yes—an infinity of them. To see why, consider what happens if each of the Li shu entries is prefixed with the words one hundred. The addition of a uniform constant to both numbers (100) and logarithms (10) means that the resulting matrix (Figure 11) will again be alphamagic.

Such a square forms an example of what I call the second harmonic of the fundamental (first harmonic) square. Using two hundred instead of one hundred would result in the third harmonic, and so on. Subharmonics ("zero point . . .") are conceivable too, if a little far-fetched. The harmonic phenomenon thus gives rise to an endless progression of alphamagic squares, none of them claiming our serious further interest (save perhaps in specialized contexts) when once their fundamentals have been identified. What about the latter?

Figure 12 presents (in numerical form) the first ten English alphamagic squares of order 3; rotations and reflections of the same square are counted identical. Alphamagics using repeated numbers I deem trivial; repetitions in their logarithm squares (shown alongside) are not. The ten are put in sequence firstly by magic constant, which for order 3 is equivalent to ranking by center number, and secondly by the lowest number occurring: 2 in the first square, 5 in the second, and so on. Extendable to higher orders, this system attaches a unique index number to every square, thus providing a convenient method of reference. Where the lowest numbers of different squares coincide, ranking will depend on the second lowest, and so on. As with ordinary magic squares, standard practice is to reproduce examples so that the smallest corner number appears in the top left-hand position, with the smaller of its two immediate neighbors oriented to the top row (middle cell). Where different squares employ identical numbers, as may occur with higher orders, this latter convention will determine rank.

Looking over the list, certain characteristic features emerge. As
intuition might have led one to surmise, No. 1, the primordial Anglo-Saxon square, being the smallest and simplest (as well as oldest) exemplar in the language, is indeed none other than the Li shu, the Arkenstone among alphamagic gems, unmatched in revealing consecutive, minimal logics. Aside from its harmonics (Nos. 17, 26, 126, . . .), we have to ascend to the 91st square (magic constants = 885;60) before finding another consecutive specimen (Figure 13). There is only one other such fundamental square among the 217 alphamagics constructible from the English number-names up to five hundred, No. 120 (magic constants = 897;60) (Figure 14).

As we see, increasing word length entails that neither of these, nor in fact any beyond, are minimal. In our language, therefore, this is the exclusive property of the Li shu. The spirit of the yew tree knew well its errand to King Mi.

Glancing next at Square No. 7 (Figure 12), one detects the essential structure underlying the formation of 3×3 alphamagic squares: the well-known mathematical structure known as the greco-latin or Eulerian square. By a latin square of order N we mean one having N² entries of N different elements, none of them occurring twice in any row or column (Figure 15). A greco-latin square is one formed by superimposing two suitable latins such that each cell becomes occupied by a distinct entry. The term greco-latin derives from the once-common practice of using Greek and Roman letters to distinguish their two components; squares of this kind were first investigated in the 1770s by the great mathematician Leonhard Euler. It is easy to prove that order 3 admits of just one possibility—the square shown as Figure 16. (In combining a pair of latins it is not essential to add their separate elements, as is done here, but merely to append the contents of corresponding cells.)

Comparing this with Square No. 7 (among others, see also Nos. 1, 3, 6, 8, 9), the identity of form is immediately apparent (you will note the correspondence: $A \leftrightarrow 4$, $B \leftrightarrow 6$, $C \leftrightarrow 5$, $a \leftrightarrow 4$, $b \leftrightarrow 1$, $c \leftrightarrow 7$). Rows and columns (but not diagonals) in numerical representations of these squares are therefore composed of different permutations of the same set of digits. I leave it to readers to show that if $a = (b+c)/2$ and $C = (A+B)/2$ (the conditions necessary for magic diagonals), the resulting matrix is isomorphic with Lucas's formula. We shall have more to discuss about greco-latins later.

Staying with Square No. 7 for a moment, observe that the distribution of 1s, 4s, and 7s in the units' position of every entry has a curious consequence. Due to the chance that $\log 1 = \log 2$, $\log 4 = \log 5$, and $\log 7 = \log 8$, adding 1 to every number in the matrix results in a second alphamagic square: No. 9. Squares Nos. 6 and 8 form a similar related dyad. There are sixteen of these pairs—some adjacent, some more widely separated—among the first 100 squares.

The alphamagic properties of Square No. 7 are not yet entirely exhausted. Although trivial, the magic (latin) square formed by its logarithms (which I shall designate by $\log \{\text{No. 7}\}$) is worth a closer look. Writing out $\log \{\text{No. 7}\}$ in full, we have Figure 17 as a result. Viewed thus, a natural question arises: could $\log \{\text{No. 7}\}$ by any chance be alphamagic, albeit trivial, too? The answer, of course, is yes, the magic constant of $\log[\log\{\text{No. 7}\}]$ being 12 (see Figure 18).

At this point it is difficult not to wonder whether this second latin (magic) square is in turn alphamagic itself. Alas, repetition of the same process yields only a semimagic derivative. Leaving apart superficial cases where the initial
logarithm square is made up of nine identical numbers (a far from uncommon occurrence), I have been unable to find any such instance among the first few hundred English squares. No. 7 shares its distinction with Nos. 5, 9, and 36.

There is an interesting computer project here that ambitious readers may like to follow up. Ideally, of course, we seek a square giving rise to an unbroken chain of alphamagic derivatives, culminating, as any chain eventually must do, in a closed loop. The shortest and most elegant such alphamagic loop would be a self-reproducing square—Figure 19. I leave more complicated loops to the contemplation of interested parties. Lest the ground to be explored here seem unduly narrow, bear in mind that we are under no compunction to remain in the same language at each stage in the derivation process. What, for instance, might be the longest chain of multilingual alphamagic links constructible? In any case, the search for ever more potent magic “spells” of this and other kinds soon encourages a glance beyond the confines of English.

Exotic Squares

The exact number of alphabetic languages used throughout the world has perhaps never been estimated. Clearly there are many. Besides those like our own employing Roman letters, there remain others using the Greek, Hebrew, and Cyrillic alphabets. The work of collecting and collating alphamagic squares in the various tongues and dialects opens a wide (if decidedly recondite) area of research. One has only to think of the enormous literature on ordinary magic squares, with its endless refinements and ramifications, almost all of which become reapplicable to alphamagic squares, to catch a glimpse of the undeveloped possibilities. My own peregrinations in the field having been superficial, I shall present here only a few examples of order 3.

Investigating $3 \times 3$ alphamagic squares in different languages calls for no alterations to the program already described, save in loading appropriate logarithm data into memory. Having had some experience in this line of late, I can report that ascertaining the correct spelling of foreign cardinals is often trickier than one suspects. Books supposedly supplying this information should be treated circumspectly (in French, is 101 cent un or cent et un?). Typing in word lengths without introducing errors is another task requiring perseverance and concentration; a subprogram for calculating letter-counts from the words themselves is advisable. Without care in this preparatory phase, interpretation of the printout is troubled with doubts.

Taking French as an initial object of study, I was intrigued to discover only a single alphamagic square using number-names in the range up to *deux cents* (200). It seemed that Gallic orthography combined with a vigesimal (twenty-based) system of counting to produce singular effects on the alphamagic plane. Thinking what a rare collector’s item this must represent if it turned out to be the sole existing French alphamagic square of order 3, I quickly extended the search up to *trois cents*, only to be glutted with a sudden deluge of 225 new specimens! Square No. 14 (magic con-
### Alphamagic Squares Around the World

<table>
<thead>
<tr>
<th>Number of alphamagic squares</th>
<th>Translations</th>
<th>Total number of translations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Danish</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Dutch</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>English</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>Esperanto</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Finnish</td>
<td>13</td>
<td>12</td>
</tr>
<tr>
<td>French</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Gaelic</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>German</td>
<td>221</td>
<td>77</td>
</tr>
<tr>
<td>Icelandic</td>
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<td>2</td>
</tr>
<tr>
<td>Indonesian</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Italian</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Latin</td>
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<td>0</td>
</tr>
<tr>
<td>Maltese</td>
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<td>9</td>
<td>2</td>
</tr>
<tr>
<td>Spanish</td>
<td>14</td>
<td>2</td>
</tr>
<tr>
<td>Swahili</td>
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<td>4</td>
</tr>
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<td>25</td>
</tr>
<tr>
<td>Welsh</td>
<td>26</td>
<td>7</td>
</tr>
</tbody>
</table>

**Figure 22.** What is the total number of alphamagic squares with cardinals not higher than 100? This chart shows the answer for squares in different languages (left). In the column marked “translations,” a circled number represents the index number of a given square, with lines linking numerically identical squares (mutual translations). Thus, Dutch Square No. 4 is a translation of German Square No. 77.

...continues...

*...stated = 336;27* was the first (of 3) to show consecutive logarithms (Figure 20). [In parentheses, the numerical representation of each number-word is followed by the log or letter-count.]

A curiosity worth remarking is the prevalence of prime numbers among French alphamagics, a by-product of frequent *un, trois,* and *sept* terminations. Even so, a square composed uniquely of primes, in this or in any other language, has yet to be identified. The urge to uncover specialist items of this kind will probably prove a stimulus to logophiles for some time to come. Serious aficionados will hardly rest until the Tower of Babel has been ransacked from roof to basement.

Following the French experience, I was better prepared for a foray into German. After entering the new logarithms and typing *RUN,* within seconds the printer chirped into life and began spitting out alphamagic squares in a steady rhythmical tattoo evocative of massed hordes on the march.
The reason for this regularity was soon apparent: every one of the 221 squares resulting from number-names under *hundert* (100) employs nine double-digit numbers; with few exceptions, the adjacentely printed logarithms of every one of these nine were the same: 14.

Many readers, I imagine, will be surprised to learn of hundreds of alphamagic squares extant in three different languages. How is this prodigality made possible? The answer lies, simply enough, in the (inevitable) regularity of our naming systems for cardinals higher than *twenty*, the designations beyond this point being exact verbal counterparts of their decimal-digit representations (*twenty-one* = 20 + 1, *twenty-two* = 20 + 2, and so on). Thus, the combinative properties of numbers are often paralleled in their logarithms, with the result that many an unexceptional magic square (of which there are myriads, contrary to expectation), is automatically rendered alphamagic. In German—an extreme case, where the words for 1, 2, 3, 4, 5, 8, and 9 all have four letters, and those for 20, 30, 40, 50, 60, 70, 80, 90, and 100 all have seven—this factor issues in a rash of uniform logarithm squares, few of them revealing any redeeming feature of interest. A typical example is No. 72, shown in Figure 21 (magic constants = 165;42).

The trouble with squares generated by this parallel effect is their structural transparency, which robs them of logological charm. As logophiles we prize cunning arrangements exploiting unsuspected linguistic fortuity. In almost any language, therefore, the vast majority of squares will fail to command admiration. In general, of course, as in Gardner's marvelous anagram prefacing this article, alphamagic elegance resides in small numbers.

Wearying of pedestrian languages, I turned next to some of the less familiar tongues. Keeping research within manageable bounds, surveys were limited to cardinals in the range up to 100. Figure 22, a recherché anthology if ever there was one, records the numbers of squares discovered in each case. Totals are generally modest, which is not to say they would remain so if the census were extended further. Raising the ceiling to 200, for instance, second harmonics will account for a doubling in figures, at the very least.

A study of squares in foreign languages can hardly proceed very far before an obvious contingency springs to thought. Has anyone noticed, I wonder, that the German square No. 72 given above is a perfect translation of English square No. 9? As a matter of fact, both the German translation of log₁₀ [No. 9] and the English translation of log₁₀ [No. 72] are themselves alphamagic, like their originals; but here we are straying into a less central, even frivolous hinterland. Once glimpsed, of course, the notion of such a (primary) correspondence soon urges systematic comparison among squares, alphamagic translations forming yet another branch to explore in the logological labyrinth.
Alphamagic Translations between Swedish and Swahili

| Swedish No. 1 | | Swedish No. 2 | | Swedish No. 3 | | Swedish No. 4 | | Swahili No. 6 | | Swahili No. 8 |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| **Fyrtiofem** | **Sextiotre** | **Femtiosju** | **Arobaini** | **Sitini** | **Hamsini** | **Arobaini** | **Sitini** |
| *(45;9)*      | *(63;9)*      | *(57;9)*      | **na tano**  | **na tatu** | **na saba** | **na tano** | **na tatu** |
| **Sextiosju** | **Femtiofem** | **Fyrtiotre** | **(67;9)*    | *(55;9)*    | *(43;9)*    | *(67;12)*    | *(55;13)*    |
| *(67;9)*      | *(55;9)*      | *(43;9)*      | **na saba**  | **na tano** | **na tatu** | **na saba** | **na tano** |
| **Femtiofem** | **Fyrtiosju** | **Sextiotre** | *(53;9)*    | *(47;9)*    | *(65;9)*    | *(53;13)*    | *(47;14)*    |
| *(53;9)*      | *(47;9)*      | *(65;9)*      | **na tatu**  | **na saba** | **na tano** | **na tatu** | **na saba** |

| Swedish No. 5 | | Swahili No. 7 |
|---------------|---------------|
| **Fyrtiofem** | **Arobaini** |
| *(55;9)*      | **na tano**  |

Figure 25. Swedish alphanagics Nos. 1, 2, and 3 translate into Swahili alphanagics Nos. 4, 6, and 8. Word-game players will note that in Swahili, *sita* (6) is an anagram of *tisa* (9).

Figure 22 includes a résumé of the interlingual connections so far established.

Two of the languages listed show no alphanagic squares at all in the range investigated. Extending examination of the first of these discovers six Danish squares using numbers below *tohundrede* (200).

Likewise, in the second case, four squares are brought to light. No. 4 (magic constants = 411;60) being a rare consecutive-logarithm curiosum using odd numbers only (Figure 23). Here the influence of underlying latin squares is unmistakable. Likewise, early Roman influence is perhaps responsible for the consecutive logarithms to be found in Figure 24, a modern Italian ("I, a Latin") square—No. 3 (magic constants = 381;45). Note the constant difference between corresponding entries at both numerical and logarithm levels in this geographically related pair.

Oddly, of all the languages so
far examined, there is one which stands out as peculiarly rich in alphamagic translations. English is poor, yielding only the example previously cited. French, together with others, has none. Norwegian and Samoan show three, as do Swedish and Swahili, an alliterative duo remarkable in that Nos. 1, 2, and 3 in the former translate into Nos. 4, 6, and 8 in the latter (as shown in Figure 25). German yields no less than four, which is not surprising in view of its total of 221 squares. And Turkish delights in five, three of them correlating with squares in the most prolific source of all: it is the language of the West Britons, the language of the Bards, Welsh.

Together with its sister tongues Breton and Cornish, Welsh belongs to the Celtic family of languages, which includes Irish, Manx, and Gaelic. In former times the vigesimal system was current, but except in reading the clock, present-day Welsh has replaced this with decimal usage. Whether the originators of this reform had any premonition of its alphamagic consequences must remain conjectural, but the effects have been remarkable indeed. Old-fashioned (vigesimal) Welsh, which I have also examined up to cant (100), manifests no alphamagic squares whatever. Modern Welsh, on the other hand, rejoices in twenty-six squares in this range, no less than eight of them corresponding to translations of squares in either Turkish (3 cases), Samoan (2 cases), Spanish, Icelandic, or Norwegian. The latter instance furnishes a striking consecutive-logarithm cameo using even numbers only (magic constants = 216;33); see Figure 26.

Amazingly, as many as six of these twenty-six Welsh squares show consecutive logarithms, a staggering total considering that of the remaining 333 squares spread over twenty languages in Figure 22, there is but a single instance of another consecutive-logarithm square: the Li shu (the French, Latin, and Italian examples given earlier lying outside our two-digit range). None of the Cambrian six are minimal, however, the shortest number-name in Welsh containing two letters, while the smallest series of logarithms occurring runs from 7 up to 15.

A detailed treatment of the unnumbered curiosities and secondary correlations to be found among alphamagic squares across the different languages is beyond the scope of a single article. Leaving the field to enthusiasts who may like to pursue these researches—seeking, perhaps, what I failed to discover, a triple-language alphamagic translation—it is time to return to English and a look at the higher orders.

What happens in going beyond 3 x 3 specimens to larger squares? Is the way strewn with gems "beyond the wildest fantasies of logomania," or are these really all "just so much logological junk"?

In Part II of this article, to appear in the next issue, investigation into the higher orders turns up unexpected challenges to ingenuity, with intriguing sidelights on "normal" alphamagics and the role of "minimal formulae" in seeking solutions. Programmers with a taste for recreations are promised rich pickings in fresh realms of opportunity.

References
